When we discuss conditional convergence, we usually note that there is really only one way — namely, using the alternating series test — that you know of that can prove that a series converges when it does not converge absolutely. But not all signed series are strictly alternating! Here is a nice example which I claim converges conditionally, but not absolutely:

$$\sum_{n=1} \frac{\sin(n)}{n}$$

That's the sine of n radians there, meaning the terms in the series change sign often, but not by alternating strictly. Also the terms do not decrease monotonically in magnitude. The first few terms are approximately

$$0.84147 + 0.45464 + 0.04704 - 0.18920 - 0.19178 - 0.04656 + 0.09385 \ldots$$

Apart from the tests we discussed — the alternating series test and a few (e.g. integral test, comparison, ratio, etc.) which can only test positive series — the only hope you have of deciding convergence, in general, is to be able to find a supple formula for the partial sums. I will do that — sort of — in this example, showing a new idea. This "trick" is used pretty often, actually, as a way of speeding up convergence of a series, for those times when you have to know what a convergent series converges *to*.

Let $b_n = \sin(1) + \sin(2) + \ldots + \sin(n)$, that is, the b_n are partial sums for a *different* series (one without denominators). We're interested in these because of a little trick: with this definition of the b_n we have $b_1 = \sin(1)$ and for all other $n > 1$ we have

$$\sin(n) = b_n - b_{n-1} = -b_{n-1} + b_n$$

We substitute this into our original series, hoping for a nice "formula" for the partial sums. And indeed, the n^{th} partial sum of our original series is

$$S_n = \frac{\sin(1)}{1} + \frac{\sin(2)}{2} + \ldots + \frac{\sin(n)}{n}$$

$$= \frac{b_1}{1} + \frac{-b_1 + b_2}{2} + \ldots + \frac{-b_{n-1} + b_n}{n}$$

$$= b_1\left(\frac{1}{1} - \frac{1}{2}\right) + b_2\left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + b_{n-1}\left(\frac{1}{n-1} - \frac{1}{n}\right) + \frac{b_n}{n}$$

$$= \frac{b_1}{1 \cdot 2} + \frac{b_2}{2 \cdot 3} + \ldots + \frac{b_{n-1}}{(n-1) \cdot n} + \frac{b_n}{n}$$

Now, what makes this trick worthwhile is that we have a good handle on the b_n's: using some trigonometric identities, one may show by induction on n that

$$b_n = \frac{\cos(\frac{1}{2}) - \cos(n + \frac{1}{2})}{2\sin(\frac{1}{2})}$$

I leave it to you to show that this expression is never more than about 2 nor less than about -0.2 (hint: substitute x for n and use Calc-I stuff, or at least look at the graph on your calculator).

This is very useful information. It assures us that as $n \to \infty$ we will have $\frac{b_n}{n} \to 0$, so the limit of our partial sums is the same as the limit of the rest of the expression above:

$$T_n := \frac{b_1}{1 \cdot 2} + \frac{b_2}{2 \cdot 3} + \ldots + \frac{b_{n-1}}{(n-1) \cdot n}$$

But what does it mean to "take the limit of T_n"? This T_n is *precisely* the n^{th} partial sum of a new series,

$$\sum_{n \geq 1} \frac{b_n}{(n+1)n} = \frac{1}{2} \cot\left(\frac{1}{2}\right) \cdot \sum \frac{1}{n(n+1)} - \frac{1}{2} \csc\left(\frac{1}{2}\right) \cdot \sum \frac{\cos(n + \frac{1}{2})}{n(n+1)}$$

so the convergence of our original series means exactly the same as the convergence of this new series; they even have the very same sum. And I have here written the new series in terms of two other series, the first of which is easily seen to converge to 1. (It's actually the telescoping series $\sum(\frac{1}{n} - \frac{1}{n+1}) = 1!$) .)

So does all this do us any good? I mean, can we tell whether this *new* series converges? After all, this new one has cosines instead of sines but looks, if anything, worse than the series we started with! But it *is* better! The key point is that the denominators are roughly as big as n^2 now, rather than n, and as you know a p-series with $p = 1$ behaves very differently from a p-series with $p = 2$, giving us something useful to use the Comparison Test on. This new series even converges absolutely, since the n^{th} term is less than $\frac{1}{(n+1)n}$, and the series $\sum \frac{1}{n(n+1)}$ converges, as noted above. So in fact every series we have written down is now seen to converge, including the one we started with. So we have accomplished half our original goal.

Now, I said at the outset that I wanted to show you how you could prove convergence for a series that only converged *conditionally*, so I should show you that this series does not converge absolutely. In other words, I want to show you that the (positive) series

$$\sum \left| \frac{\sin(n)}{n} \right|$$

diverges (to infinity). It's easy to understand why this must be true; it's a little harder to get all the details right. In spirit, all you need to do is to point to every (roughly) 6th term and you will see they're all pretty large — large enough that their sum explodes.

More precisely, consider the sequence consisting of the integers closest to $(2k + \frac{1}{2})\pi$: the integers $2, 8, 14, 20, 27, 33, 39, 46, 52, 58, 64, \ldots$. Any two consecutive integers differ by 1 (duh!) so when I refer to "the integer closest to x", that's a number which is between $x - \frac{1}{2}$ and $x + \frac{1}{2}$. So the k^{th} term in this sequence is no more than $(2k + \frac{1}{2})\pi + \frac{1}{2} = (2\pi)k + (\pi + 1)/2 < (2\pi)(k + 1)$.

On the other hand these integers are chosen to be near the places where the sine function equals 1. Just past the peak, the sine function is decreasing (duh), so $n < (2k+\frac{1}{2})\pi + \frac{1}{2}$ implies $\sin(n) > \sin((2k+\frac{1}{2})\pi + \frac{1}{2}) = \cos(\frac{1}{2})$. Almost identical reasoning proves the same inequality when n is just to the left of the peak.

So the sum of the 2^{nd}, 8^{th} , 14^{th} , ... terms of the sequence is more than

$$\sum_{k\geq 1} \frac{\cos(\frac{1}{2})}{(2\pi)(k+1)}$$

which is a constant multiple of the harmonic series, and hence divergent. The sum of ALL the terms of $\sum \left|\frac{\sin(n)}{n}\right|$ is obviously going to be even larger, so that one diverges too.

Thus the original series does NOT "converge absolutely". Since it DOES converge, we say it "converges conditionally".

You may object that I used a "trick" to prove the original series converges. Any trick that we use more than once in our mathematical lives we call a "tool" :-) . This one is called "Summation By Parts". It's akin to Integration By Parts. Remember what that procedure lets you do for integrals: if your integrand is a product of two functions, one easy to differentiate and another easy to antidifferentiate, then you can re-express your original integral in terms of another integral; the new integrand is the product of the derivative of the first function and the antiderivative of the second. Well, replace "function" by "sequence", "derivative" by "sequence of successive differences $a_n - a_{n-1}$", and "antiderivative" by "sequence of partial sums",* and you'll see that the transformation I performed, taking me from the original sequence to the new one, is just an Integration-By-Parts look-alike. You can apply this trick over and over and get other sums of the form

$$\sum_{n\geq 0} \frac{\sin(n)}{n(n+1)\ldots(n+k)}$$

for larger and larger k; these converge really quickly when k is large.

And using this tool, we can estimate the numerical value of the infinite sum: if you add up the first N terms of the new series by hand, that is, if you calculate the N^{th} partial sum T_N, you will have a good estimate for the sum of the whole *infinite* series. Indeed, all the terms you left out add up to less than $\sum_{n>N} \frac{2}{N(N+1)}$, which we can estimate (by several of the tests) to be less than $2/N$. So you can for example add the first 100 terms by hand and know that the real sum differs from this by no more than $2/100 = 0.02$.

If you combine the last two paragraphs you get a fast way to sum the original series, applying Summation By Parts multiple times and then computing partial sums by hand. It's not too hard to deduce that the sum of this infinite series must be approximately 1.0707963 (with all of those digits correct).

* You should note that these replacements for the operations of differentiation and anti-differentiation really are inverses of each other!

So we have analyzed a particular series, which is neither positive nor alternating: we have concluded it converges (but not absolutely), and we have estimated the sum. In fact, it can be shown that this series converges to

$$\frac{\pi - 1}{2}$$

Formally this is easy to derive: use the fact that $\ln(1-u) = -\sum u^n/n$ with $u = e^{ix}$ and $u = e^{-ix}$; subtract the two resulting expressions (using DeMoivre's Formula) and divide by $2i$ to get a closed-form expression that equals $\sum \frac{\sin(nx)}{n}$, but which on the other hand works out to $\frac{\pi - x}{2}$. But making sense of these manipulations is tricky because the logarithm is not well defined for complex numbers, and moreover these series only converge appropriately when x has a non-zero imaginary part. Indeed, the results we obtained this way are only valid if $x < 2\pi$; certainly the sum should not change its value when x is increased by a multiple of 2π! Nonetheless, they point towards a method that can be used, and they propose a sum which agrees with what we can compute numerically.

If you pursue more math courses as far as Fourier Analysis, you will eventually see other kinds of series besides the ones we discuss in class. Just as we have used Taylor series $\sum a_n x^n$ to "describe" "any" function (near $x = 0$), one may use Fourier series $\sum a_n \sin(nx) + b_n \cos(nx)$ to "describe" "any" periodic function. The function f which is defined to be $f(x) = (\pi - x)/2$ when $0 < x < 2\pi$ and periodic outside that interval (i.e. $f(x + 2\pi) = f(x)$ for all x) — a function with a "sawtooth" graph — has a Fourier series with each $a_n = 1/n$ and each $b_n = 0$. This observation allows us to evaluate the series $\sum \frac{\sin(nx)}{n}$, and more importantly points to the significance of signed series in general.