BENNETT LINEAR ALGEBRA EXAM — ANSWERS Dec 12 2017

1. Find the four real numbers a_i for which

$$\frac{8x+12}{(x^2-1)^2} = \frac{a_1}{x-1} + \frac{a_2}{(x-1)^2} + \frac{a_1}{x+1} + \frac{a_2}{(x+1)^2}$$

ANSWER: We are looking for four real numbers making

$$a_1(x-1)(x+1)^2 + a_2(x+1)^2 + a_3(x-1)^2(x+1) + a_4(x-1)^2 = (8x+12)$$

as functions of x. We must determine a set of four equations that will allow us to solve for these unknowns. Among the options: (a) If this equation is to hold for all x then it must in particular be true when $x = -1, 0, 1, 2$, giving the equations

$$4a4 = 4, -a1 + a2 + a3 + a4 = 12, 9a1 + 9a2 + 3a3 + a4 = 28, 32a1 + 16a2 + 16a3 + 4a4 = 36$$

Or (b) we can equate coefficients on the two sides of the equation, giving

$$-a1 + a2 + a3 + a4 = 12, -a1 + 2a2 - a3 - 2a4 = 8, a1 + a2 - a3 + a4 = 0, a1 + a3 = 0$$

Or (c) we can insist that the two sides have equal derivatives of all order at say $x = 1$:

$$4a2 = 20, 4a1 + 4a2 = 8, 8a1 + 2a2 + 4a3 + 2a4 = 0, 6a1 + 6a3 = 0$$

Or (d) mix and match these ideas, e.g. evaluating the equation and its first derivative at both $x = \pm 1$:

$$4a2 = 20, 4a1 + 4a2 = 8, 4a4 = 4, 4a3 - 4a4 = 8$$

All these sets of equations are satisfied by the same quadruple of numbers: $a_1 = -3, a_2 = 5, a_3 = 3, a_4 = 1$.

The expression on the right side is called the "Partial Fractions Decomposition" of the function on the left side. The ideas above are used to compute it, but Linear Algebra is used to prove it exists in the first place!

2. Suppose A and B are square matrices of the same size, and that $AB = 0$. Must $BA = 0$ too? (If you say "yes", give a proof; if you say "no", give a counterexample.)

ANSWER: No, for example if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $AB = 0$ and $BA = A \neq 0$.

3. Compute $\det(C)$ where C is the $n \times n$ matrix with $C_{ij} = 1$ if $i \neq j$ and $C_{ii} = 0$.

ANSWER: You can expand the determinant across the top row (say) to get an alternating sum of $n-1$ determinants of $(n-1)\times(n-1)$ matrices; in each of these the first column is all 1's. Subtracting that column from every other column shows that each of these matrices has a determinant of ± 1, and with some attention to signs we deduce $\det(C) = (-1)^{n-1}(n-1)$. Other sequences of row-operations are useful, e.g. adding all the other rows to the top row, pulling out a factor of $n-1$ from that row, then subtracting that row from all the other rows.

Here's an alternative approach: $K = C + I$ is the matrix filled with 1's so we trivially have $K^2 = nK$. This means that all the eigenvalues of K satisfy $\lambda^2 = n\lambda$: either $\lambda = n$ or $\lambda = 0$. In fact there are $n-1$ obvious, linearly-independent eigenvectors with $\lambda = 0$: $v_1 = (1, -1, 0, 0, \ldots), v_2 = (0, 1, -1, 0, 0, \ldots), \cdots$, and $v_{n-1} = (0, \ldots, 0, 1, -1)$. An eigenvector for $\lambda = n$ is also easy to spot: $v_n = (1, 1, 1, 1, 1)$. Thus K is diagonalizable — in the basis of these eigenvectors, K is the diagonal matrix with $(n-1)$ 0's and one n on its diagonal. Thus we know its characteristic polynomial of K is $\det(xI - K) = x^{n-1}(x-n)$. Well then, set $x = 1$ to see $\det(C) = (-1)^n \det(-C) = (-1)^n \cdot 1^{n-1}(1-n) = (-1)^{n-1}(n-1)$.

(You could instead observe that these n vectors are eigenvectors for C, too, so that C is diagonalizable and its determinant is the product of the eigenvalues, $(n-1), -1, -1, -1$, and -1.)

4. Suppose M is the 3×3 matrix which represents a $180°$ rotation around the line $x = y/2 = z/3$. (That's the line that contains the vector $\langle 1, 2, 3 \rangle$.) What are the eigenvalues of M? For extra credit give also the eigenvectors.

ANSWER: That line stays fixed so e.g. $\langle 1, 2, 3 \rangle$ is an eigenvector with eigenvalue 1. Everything in the plane perpendicular to this line is rotated 180 degrees, i.e. is sent to its own negative, so they are all eigenvectors with eigenvalue -1. For example, $\langle -2, 1, 0 \rangle$ and $\langle -3, 0, 1 \rangle$ are linearly independent vectors in this space, and therefore span the (-1)-eigenspace. So the eigenvalues are ± 1 with the eigenspaces being the spans of these vectors.

There is no need to compute the matrix C itself but if you want it you may have it! From the information above we have $C = EDE^{-1}$ where E is the matrix whose columns are the three eigenvectors and D is the diagonal matrix with diagonal entries $1, -1, -1$. I make this out to be

$$\begin{pmatrix} -6/7 & 2/7 & 3/7 \\ 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & 2/7 \end{pmatrix}$$

5. Suppose V is the vector space of all 3×3 matrices. Let $\mathcal{L}$ be the set of linear maps from V to V. This $\mathcal{L}$ is a vector space (you don't have to prove that).

(a) Show that for every invertible 3×3 matrix P, the function $f : V \to V$ given by $f(M) = PMP^{-1}$ is in $\mathcal{L}$ (i.e. show that f is a linear transformation).

(b) Are there other elements of $\mathcal{L}$ besides those in (a)? (If you say "no", give a proof. If you say "yes", find one.)

ANSWER: For (a) we need only check the two parts of the definition of a linear transformation. If M and N are matrices we may use the distributive property of matrix multiplication twice to show $f(M+N) = P(M+N)P^{-1} = (PM+PN)P^{-1} = PMP^{-1} + PNP^{-1} = f(M)+f(N)$ and similarly if c is any real number then $f(cM) = P(cM)P^{-1} = c(PM)P^{-1} = c\,f(M)$.

More interesting is part (b). Since V is a nine-dimensional vector space, the set of linear maps from V to V can be identified with the set of 9×9 matrices, which is an 81-dimensional vector space. (You can also prove this by writing out what the 9 components of a matrix $f(M)$ are, in terms of the 9 entries of M itself; each is a linear function of these 9 variables, so each requires 9 separate coefficients to write it out, giving 81 coefficients necessary to describe the whole linear map.) The collection of linear maps given in part (a) is not a linear subspace of $\mathcal{L}$ but rather a more curvy geometric object; in an appropriate sense it's only 8-dimensional, meaning the transformations in (a) are actually a very small part of $\mathcal{L}$.

So it's not hard to find other elements of $\mathcal{L}$ but actually verifying that they are not one of the ones in part (a) is tricky. You have to find something special about *those* linear maps that is not shared by *all* linear maps.

For example, if f is defined as in part (a) then we notice that f happens to be invertible: we can define another element g of $\mathcal{L}$ to be $g(M) = P^{-1}MP$ and then observe $f \circ g$ and $g \circ f$ are both the identity map from V to itself. By contrast the zero element of $\mathcal{L}$ (that's the linear transformation sending every element of V to the zero matrix) is clearly not one-to-one and hence not invertible.

Or we could observe that if f is defined as in part (a) then f happens to "preserve products", i.e. $f(MN) = f(M)\,f(N)$ for any two matrices M and N. The transpose map $f(M) = M^t$ doesn't have this feature. It's easily shown to be in $\mathcal{L}$ but if M and N are any two matrices that don't commute with each other then $f(MN) = (NM)^t = N^t M^t \neq M^t N^t = f(M)\,f(N)$, so the transpose map is not one of the ones constructed in part (a).

For another example, notice that the maps f in part (a) all "preserve traces", that is, Tr($f(M)$)=Tr(M), because for any two matrices A, B it's true that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ (both of them turn out to be $\sum_{i,j} A_{ij}B_{ji}$) and so

$$\mathrm{Tr}(f(M)) = \mathrm{Tr}(PM \cdot P^{-1}) = \mathrm{Tr}(P^{-1} \cdot PM) = \mathrm{Tr}(M)$$

But there are certainly linear transformations that do not preserve traces, e.g.

$$F(\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}) = \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$$

A very general family of elements of $\mathcal{M}$ is the set of maps of the form $f(M) = PMQ$ where P and Q are any fixed 3×3 matrices. These include the maps in part (a) but include many others as well; in fact it is not hard to show that every element of $\mathcal{L}$ can be written as a sum of a few of these.