BENNETT DIFFERENTIAL EQUATIONS EXAM 2018 — ANSWERS

1. Find the general solution of $x^4y'' + 5x^3y' + 4x^2y = 1$.

ANSWER: We first solve the corresponding homogeneous equation, a type called a Cauchy-Euler equation, which suggests a solution of the form $y = x^r$. This is indeed a solution as long as $0 = r(r-1) + 5r + 4 = (r+2)^2$. So $y_0 = x^{-2}$ is the one solution of this type.

To get another independent solution to the homogeneous equation, and also to find a solution to the inhomogeneous equation, we will use variation of parameters, writing $y = y_0(x)v(x)$. We can simplify

$$Ay'' + By' + Cy = (Ay_0'' + By_0' + Cy_0)v + (2Ay_0' + By_0)v' + Ay_0v''$$

which reduces to just $(2Ay_0' + By_0)v' + Ay_0v''$ because y_0 is a homogeneous solution. We then let w be v' in what follows.

So in order to have another solution to the homogeneous equation we need $0 = (2Ay_0' + By_0)w + Ay_0w' = (-4x+5x)w + x^2w' = x(w + xw') = x(xw)'$ This requires xw to be constant, say $w = 1/x$; then since $w = v'$ we must have $v = \ln(x)$. So a second, independent, solution to the homogeneous equation is $y_1(x) = x^{-2}\ln(x)$.

Likewise to have a solution to the original inhomogeneous equation we need $1 = (2Ay_0' + By_0)w + Ay_0w' = (-4x + 5x)w + x^2w' = x(w + xw') = x(xw)'$. This requires $(xw)' = 1/x$, say $xw = \ln(x)$, and so $w = \frac{1}{x}\ln(x)$. Again since $w = v'$ this means $v = \int \frac{1}{x}\ln(x)\,dx = \frac{1}{2}(\ln(x))^2$. We conclude that a solution to the inhomogeneous equation is $y_p = x^{-2} \cdot \frac{1}{2}(\ln(x))^2$.

So the general solution to the original equation is $y = \left(a + b\ln(x) + \frac{1}{2}\ln(x)^2\right)/x^2$.

One might also look for these solutions among the solutions of the *homogeneous* ODE formed by differentiating the original one w.r.t. x. That one's characteristic polynomials is $(r+2)^3$, so we know our y_p is of the form $x^{-2}(a + b\ln(x) + c\ln(x)^2)$.

2. Sketch the solution to the differential equation

$$\frac{dy}{dx} = y^4 + 4 \qquad y(3) = 0$$

Identify any critical points and inflection points, and explain why there are or are not any horizontal or vertical asymptotes.

ANSWER: The graph looks vaguely like a shifted branch of the tangent function.

At a critical point we would have $dy/dx = 0$ but $y^4 + 4 > 0$ for all y, so there are no critical points, and in fact we learn that the function is everywhere-increasing.

At an inflection point we would have $0 = d^2y/dx^2 = \frac{d}{dx}y' = \frac{d}{dx}(y^4+4) = (4y^3)\frac{dy}{dx} = (4y^3)(y^4+4)$. This is zero iff $y = 0$. We do know that $y = 0$ when $x = 3$ but since y is everywhere increasing, there can be no other values of x for which $y(x) = 0$. Not only is the point $(3, 0)$ a *candidate* to be an inflection point, but indeed the concavity does change there: $y'' > 0$ iff $y > 0$ and since y is increasing, this will be true for all $x > 3$; for all points with $x < 3$ the function is instead decreasing.

Since $y' = y^4 + 4 > 4$ it follows that $y(x) > 4(x-3)$ for all $x > 3$. (For example, the graph will pass $y = 1$ before we go as far as $x = 3\frac{1}{4}$.) In particular, this inequality is inconsistent with having a finite limit as $x \to +\infty$, so there is no horizontal asymptote on the right. The same reasoning shows there is none on the left either.

There are however vertical asymptotes. We have already noted that we will have $y(x_0) = 1$ for some $x_0 < 3.25$, with y continuing to increase when $x > x_0$. Then since $y > 1$, we have $y^4 + 4 > y^2 + 4$. It follows that for all larger values of x, the values of $y(x)$ will be larger than those of $z(x)$, where z is the solution to the equation $z' = z^2 + 4, z(x_0) = 1$. This equation is easily solved: $z = 2\tan(2x + C)$ for some constant C. Clearly this function has a vertical asymptote between $x = x_0$ and $x = x_0 + \pi/4$, that is, $z(x) \to +\infty$ somewhere in that interval. Since $y(x) > z(x)$ it follows that $y(x) \to +\infty$ in that interval as well. (Similar observations show $y(x) \to -\infty$ in a short interval to the left of $x = 3$.)

If you wish, you may solve for $y(x)$ explicitly. This is a separable differential equation, and the required integration is that of a rational function of y; you can get a closed form using Partial Fractions. (Hint: $y^4 + 4 = (y^2 + 2y + 2)(y^2 - 2y + 2)$.) The result is

$$16(x-3) = \log(1 + 4y/(y^2 - 2y + 2)) + 2(\arctan(y+1) + \arctan(y-1))$$

so if we let $y \to \pm\infty$ we see the vertical asymptotes are at $x = 3 \pm \pi/8$

The substitution $y = \sqrt{2\tan(x)}$ converts the rational integral here into precisely the integral $\int \sqrt{\tan(x)}\, dx$ involved on this day's Calculus contest!

This question was inspired by one taken from the Math GRE exam.

3. Solve the differential equation

$$(4xy + 2y^2 + 2x)\frac{dy}{dx} = x^2 + 2xy + 3y^2 + 2y \qquad y(1) = -2$$

Hint: there is an integrating factor μ for which $\partial\mu/\partial x = \partial\mu/\partial y$.

ANSWER: The point of an integrating factor μ is that it rewrites a differential equation $M\,dx + N\,dy = 0$ in an equivalent form $(\mu M)\,dx + (\mu N)\,dy = 0$ which is now exact, i.e. where now $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$. This expands to a PDE that we must solve if we hope to find μ:

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

Ordinarily we hope to find such a μ which has an additional feature like $\mu_x = 0$ or $\mu_y = 0$; here we are prompted to try finding a μ for which $\mu_y = \mu_x$. In that case, the equation to be solved has a more promising form $\mu_x(N - M) + \mu(N_x - M_y) = 0$.

In our case we begin with $M = -(x^2 + 2xy + 3y^2 + 2y)$ and $N = (4xy + 2y^2 + 2x)$ and so the equation we must solve to find μ is

$$(x^2 + 6xy + 5y^2 + 2x + 2y)\mu_x + 2(x + 5y + 2)\mu = 0$$

Fortunately the first coefficient is a multiple of the second; if we divide by $x + 5y + 2$ this equation becomes simply $(x + y)\mu_x + 2\mu = 0$, which has a solution $\mu = (x + y)^{-2}$ that does indeed have $\mu_x = \mu_y$, as promised.

So we have found an integrating factor, which means that this is now an exact form of the original ODE:

$$\frac{(x^2 + 2xy + 3y^2 + 2y)}{(x + y)^2}dx + \frac{-(4xy + 2y^2 + 2x)}{(x + y)^2}dy = 0$$

The entire left side should be the differential of a function Φ which we should find by (say) integrating (with respect to x) the coefficient of dx. That coefficient is $1 + 2y(y + 1)/(x + y)^2$, so we try $\Phi = x - 2y(y + 1)/(x + y) = (x^2 + xy - 2y^2 - 2y)/(x + y)$ and indeed Φ_y turns out to be the coefficient of dy in the exact ODE.

So the solution curves of the original equation are all the curves where Φ is constant: $(x - C)(x + y) - 2y(y + 1) = 0$. The curve passing through $(x, y) = (1, -2)$ is the one where $\Phi = 5$, which is the hyperbola $(x + 2y - 1)(x - y - 4) = 4$.

4. Solve the system $\quad \frac{dx}{dt} = y(x + y)^5, \quad \frac{dy}{dt} = x(x + y)^5, \quad x(0) = 1, \; y(0) = 0$ (*Hint:* Add and subtract.)

ANSWER: Add to discover that $u = x + y$ satisfies $du/dt = u^6$ and $u(0) = 1$; thus $u(t) = (1 - 5t)^{-1/5}$. Then the original system is more simply written

$$x' = y/(1 - 5t), \qquad y' = x/(1 - 5t).$$

Now subtract these equations to see that $v = x - y$ has $v' = -v/(1-5t)$, which is separable: $\log(v) = \log(1-5t)/5 + C$ and $C = 0$ from the initial condition. Thus $v = (1-5t)^{1/5}$. Adding and subtracting gives

$$x = \frac{1}{2}\left((1-5t)^{-1/5} + (1-5t)^{1/5}\right) \qquad y = \frac{1}{2}\left((1-5t)^{-1/5} - (1-5t)^{1/5}\right)$$

One could also profitably divide the ODEs in the original system and integrate, to discover that $x(t)^2 - y(t)^2 = 1$ (i.e. the solution parameterizes a hyperbola) so once $u = x + y$ is known we can divide to compute $v = x - y$ quickly.

This problem appeared on the 1953 Putnam competition (II-3).

5. The *biharmonic equation* from continuum mechanics is the fourth-order linear partial differential equation $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$. For partial credit, find a nonzero solution $u(x,y)$ to this equation. For full credit, find a non-polynomial solution. For extra credit, find an infinite-dimensional vector space of solutions.

ANSWER: Many types of answers are possible!

From the wording of the question you might try a polynomial for partial credit; anything of total degree less than 4 would work.

To get more solutions you might try Separation of Variables. Suppose there is a solution of the form $u = X(x)Y(y)$. Such a u really is a solution iff

$$\frac{X''''}{X} + 2\frac{X''}{X}\frac{Y''}{Y} + \frac{Y''''}{Y} = 0$$

You could for example choose X and Y so that each is a multiple of its second derivative: if $X'' = aX$ and $Y'' = bY$ then $X'''' = a^2 X$ and likewise for Y, so we have a solution as long as $a^2 + 2ab + b^2 = 0$, i.e. $a = -b$. Taking for exampls $a = 1$ and $b = -1$ then leads to solutions like $e^x \cos(y)$, and more generally $e^{kx}\cos(ky)$ (and $e^{kx}\sin(ky)$). These functions, for different values of k, are linearly independent.

(One may show that at least one of X''/X and Y''/Y *must* be constant, from which it is possible to show $X(x)Y(y)$ will be very nearly equal to one of the products found in the previous paragraph.)

Changing coordinates from x and y to $z = x + iy$ and $w = x - iy$, the biharmonic equation becomes simply $u_{zzww} = 0$! In this way we can show that the most general solution is $u = A(z) + B(w) + C(z)w + D(w)z$ for arbitrary smooth functions A, B, C, D of one variable.

Alternatively one may write the original PDE as $H(H(u)) = 0$ where H is the *harmonic* differential operator $H(f) = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$. So among the solutions to our problem are all the functions $u = f(x,y)$ for which $H(f) = 0$; one could get additional solutions by solving say $H(f) = x$, It is also reasonable to view H^2 as a linear map on the spaces of homogeneous polynomials of each degree N, and then piece these together to get power-series solutions for u.