BENNETT LINEAR ALGEBRA PRIZE EXAM Dec 11 2018

1. Find a polynomial $f(x)$ which has the same values as $g(x) = \frac{120}{x}$ for $x = 1, 2, 3, 4, 5$. (That is, we need $f(1) = 120$, $f(2) = 60$, etc.)

ANSWER: The polynomial

$$f(x) = x^4 - 15x^3 + 85x^2 - 225x + 274$$

will do. Any other solution must differ from this f by a polynomial that vanishes at all five of these points, i.e. the other solutions are precisely of the form $f(x) + (x-1)(x-2)(x-3)(x-4)(x-5)P(x)$ for any other polynomial P. In particular, this f is the only solution of degree less than 5.

To find this f, one may look for the right coefficients in $f(x) = a + bx + cx^2 + dx^3 + ex^4$ (say); since $f(1) = 120$ we have $a + b + c + d + e = 120$, and from each of the other four data points we obtain another equation. Solve that system of 5 equations in 5 unknowns, e.g. by row-reduction, in order to determine that $a = 274, b = -225$ etc.

Incidentally the coefficient matrix here is called a *Vandermonde* matrix: $M_{ij} = (x_i)^{j-1}$. The determinant of M is easily computed but here we would presumably need the inverse of M, which is not so easily found.

A simpler solution would use a different basis of $\mathcal{P}_4$, the space of all quartic polynomials. Rather than the standard basis $\{1, x, x^2, x^3, x^4\}$ we may use the five polynomials $f_i(x) = (x-1)(x-2)(x-3)(x-4)(x-5)/(x-i)$. Each $f_i(x)$ vanishes at all of the points $\{1, 2, 3, 4, 5\}$ except $x = i$ itself, where it has the value $(i-1)!(5-i)!(-1)^{5-i} \neq 0$. Thus these f_i are linearly independent and hence do indeed form a basis for $\mathcal{P}_4$. We then obtain the polynomial desired as

$$\begin{aligned} f &= 120 \cdot \frac{f_1}{24} + 60 \cdot \frac{f_2}{-6} + 40 \cdot \frac{f_3}{4} + 30 \cdot \frac{f_4}{-6} + 24 \cdot \frac{f_5}{24} \\ &= (x-1)(x-2)(x-3)(x-4)(x-5)\left(\frac{5}{x-1} - \frac{10}{x-2} + \frac{10}{x-3} - \frac{5}{x-4} + \frac{1}{x-5}\right) \end{aligned}$$

This is a typical example of *interpolation* or *curve-fitting*. Since in this problem the x_i are evenly spaced, the most practical method of hand-calculation is to first compute $f^*(x) = f(x+1) - f(x)$ from its desired values, then use f^* to deduce f. Iterating this idea, we successively compute $f^{****} = 24$, then $f^{***} = 24x - 54$, f^{**}, f^*, and then f.

2. Suppose A and B are square matrices of the same size, and that $ABABA = I$.

(a) Explain why A is invertible.

(b) Show that $AB = BA$.

ANSWER: Of *course* A is invertible: its inverse is $BABA$, since we are given that $A(BABA) = I$! (One may also note that $\det(A)^3 \det(B)^2 = \det(ABABA) = \det(I) = 1$ and so $\det(A)$ must be nonzero, so that A is invertible.)

For (b) you could simply compute

$$AB = ABI = AB(ABABA) = ABABABA = (ABABA)BA = IBA = BA$$

Or, note that $(AB)(ABA) = I$ means AB is the "left inverse" of ABA, while $(ABA)(BA) = I$ means BA is the "right inverse" of ABA; since (for matrices) left-inverses and right-inverses are equal, we have $AB = BA$. Or use $A(BA)(BA) = I$ to note that $A = (BA)^{-2}$; since a matrix commutes with its every power, BA commutes with A, i.e. $A(BA) = (BA)A$; now right-multiply both sides of this equation by A^{-1} to get $AB = BA$.

This problem was inspired by problem A4 of the 2018 Putnam Exam.

3. The exponential function is defined for square matrices A by the usual power series:

$$e^A = I + A + \frac{1}{2}A^2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

Compute e^A when $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$.

ANSWER: We can show, either algebraically or geometrically, that

$$B := e^A = \begin{pmatrix} 2e^2 - e^3 & e^3 - e^2 \\ 2e^2 - 2e^3 & 2e^3 - e^2 \end{pmatrix}$$

We will first need the eigenvalues of A. I compute

$$\begin{aligned} \det(A - xI) &= x^2 - 5x + 6 = (x-2)(x-3) \\ \ker(A - 2I) &= \ker \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \text{span}(v_2) \\ \ker(A - 3I) &= \ker \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \text{span}(v_3) \end{aligned}$$

Now we can proceed algebraically: from the data above we deduce that $A = PDP^{-1}$ where $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is diagonal and $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ (so that $P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$). But from $A = PDP^{-1}$ we deduce (by induction) that $A^n = PD^nP^{-1}$ for all $n \geq 0$, then that $c_nA^n = P(c_nD^n)P^{-1}$, and then (using the distributive properties of matrix multiplication) that $\sum c_nA^n = P\left(\sum c_nD^n\right)P^{-1}$. That is, we can evaluate e^A, or indeed any power series $f(A)$, by evaluating it on D and then conjugating by the matrix P.

Since D is diagonal, though, it is very easy to evaluate a power series: $f(D)$ will again be a diagonal matrix, and the ith entry on the diagonal will be $f(D_{ii})$. In our case that means $e^D = \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix}$. Then

$$e^A = P(e^D)P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2e^2 - e^3 & e^3 - e^2 \\ 2e^2 - 2e^3 & 2e^3 - e^2 \end{pmatrix}$$

Or we may proceed geometrically. If v is an eigenvector of A, then $Av = \lambda v$ so $A^2v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$ and more generally $A^n v = \lambda^n v$. Then we may evaluate the product of the matrix $B = e^A$, or indeed any power series $f(A)$, with the vector v: $(\sum c_nA^n)v = \sum c_n(A^nv) = \sum c_n(\lambda^n v) = f(\lambda)v$. So in our case, $Bv_2 = e^2v_2$ and $Bv_3 = e^3v_3$. Now, it's easy to write the standard basis vectors in terms of v_2 and v_3:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2v_2 - v_3, \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = v_3 - v_2$$

so we compute the effect of B on them:

$$Be_1 = B(2v_2 - v_3) = 2(e^2v_2) - (e^3v_3) = (2e^2)\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2e^2 - e^3 \\ 2e^2 - 2e^3 \end{pmatrix}$$

Be_2 is computed similarly. Then the matrix B representing this linear transformation consists of adjoining these two columns together, giving the matrix shown above.

One student also noted that $C = A - 2I$ has the property that $C^2 = C$ and so from the definition of e^C one may directly compute $e^C = I + (e-1)C$. As noted above it is also trivial to exponentiate diagonal matrices so in particular $e^{2I} = (e^2)I$. One might also hope that $e^{C+2I} = e^Ce^{2I}$, and this *is* true, but only because C and $2I$ commute. But with this caveat one then has $e^A = e^Ce^{2I} = (I + (e-1)C)(e^2I) = e^2(3I - A) + e^3(A - 2I)$.

The exponential map for matrices is very useful in the study of Lie Groups.

4. A linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ is called a *projection* if $L(L(v)) = L(v)$ for each $v \in \mathbf{R}^n$. For example the function $L(x, y, z) = (2y + 3z, y, z)$ is a projection in $\mathbf{R}^3$.

Show that the only possible eigenvalues of a projection L are 0 and 1.

ANSWER: Suppose λ is an eigenvector of L. Then there is a nonzero vector v with $L(v) = \lambda v$. Then we compute

$$L(L(v)) = L(\lambda v) = \lambda L(v) = \lambda^2 v$$

So if L is a projection then $L(L(v)) = L(v)$, so these computations force $\lambda^2 v = \lambda v$, i.e. $(\lambda - 1)(\lambda - 0)v = 0$. Now, in an vector space a scalar multiplication cv can only yield the zero vector if c is the number 0 or if v is the zero vector. But since our v is an eigenvector, it is nonzero. This forces the scalar to be 0: $(\lambda - 1)(\lambda - 0) = 0$, so that either $\lambda = 1$ or $\lambda = 0$.

5. Find an invertible matrix P for which $PAP^{-1} = B$ where

$$A = \begin{pmatrix} 1 & 2018 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 41 \\ 0 & 1 \end{pmatrix}$$

ANSWER: We need P to be invertible such that $PA = BP$. Writing $P = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, we need

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2018 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 41 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad \text{i.e.,}$$

$$\begin{pmatrix} x & 2018x + y \\ z & 2018z + w \end{pmatrix} = \begin{pmatrix} x + 41z & y + 41w \\ z & w \end{pmatrix}$$

The upper-left entries force $z = 0$, at which point all four entries will match as long as $2018x - 41w$. (y is arbitrary.) The simplest solution is $x = 41, w = 2018$, so

$$P = \begin{pmatrix} 41 & 0 \\ 0 & 2018 \end{pmatrix}$$

which is indeed invertible, so this P will suffice.