BENNETT DIFFERENTIAL EQUATION PRIZE EXAM May 11 2021

Submit your solutions *with all work shown*, **by 8pm (Austin time) as an email attachment to rusin@math.utexas.edu. During the exam you must abide the rules previously sent via email.**

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1. Two functions f and g are said to be *inverses of each other* if $f(g(t)) = t$ for every t in some interval, and $g(f(u)) = u$ for every u in another interval. For example the logarithm and exponential functions are inverses of each other, as are the cosine function and any branch of the arc-cosine function.

Find a differentiable function f for which f and f' are inverses of each other.

ANSWER: We need a function whose derivative is also its inverse. We might look in a family of functions that is closed under both operations of differentiation and inversion; one such family is the set of power functions $f(x) = Ax^B$. Such a function has derivative $A_2 x^{B_2}$ and inverse $A_3 x^{B_3}$ where $A_2 = (AB), B_2 = (B-1), A_3 = A^{-1/B}, and B_3 = 1/B$. So we simply need numbers A, B for which $AB = A^{-1/B}$ and $B - 1 = 1/B$. The latter condition is satisfied iff $B^2 - B - 1 = 0$, whose roots are the Golden Ratio $\phi = (1+\sqrt{5})/2$ and its negative reciprocal $\bar{\phi} = (1-\sqrt{5})/2$. In either case we additionally need $A^{1+(1/B)} = 1/B$, which simplifies to $A = (1/B)^{1/B}$. Numerically, our function is approximately $0.7427429447x^{1.618033988}$.

2. Find the general solution of $yy'' + (y')^2 = 5(y')^3$

ANSWER: This equation is second-order autonomous, so we can reduce the order with the substitution $v = y'$, since then $y'' = v' = v\,\frac{dv}{dy}$ and the original equation implies $y\,v\,\frac{dv}{dy} + v^2 = 5v^3$. This equation holds if $v = 0$, i.e. if y is any constant; otherwise we have $\frac{dv}{dy} + \frac{1}{y}v = \frac{5}{y}v^2$. This is now a Bernoulli differential equation (with exponent 2), which we can solve by dividing by v^2, to get an equation which is first-order linear in $u = v^{-1}$. Indeed, we have $du/dy = -v^{-2}\,dv/dy = \frac{1}{y}v^{-1} - \frac{5}{y} = \frac{1}{y}u - \frac{5}{y}$. Now we use an integrating factor of $1/y$ to get

$$\frac{d(u/y)}{dy} = -5/y^2 = \frac{d}{dy}(5/y)$$

so that $u/y = 5/y + C$ for some C. Solve for u and recall $v = 1/u = 1/(5 + Cy)$. But $v = dy/dx$ so we end up with a separable differential equation in x and y: $(5+Cy)dy = dx$

The general solution is $x = 5y + (C/2)y^2 + C'$. When $C = 0$ we get the solutions $y = \frac{1}{5}(x - x_0)$ for arbitrary constant x_0; otherwise we may use the Quadratic Formula to get solutions which may be written

$$y = \frac{-5 \pm \sqrt{2c(x - x_0)}}{c}$$

with arbitrary x_0 and nonzero c. (I am writing the solutions in terms of $x - x_0$ since the original equations were independent of x.)

3. An object moves along the real number line; its position $x(t)$ at time t satisfies

$$x''(t) + 4x(t) = x(t)\,(x'(t))^2$$

Show that if $x(0) = 1$ and $x'(0) = 0$ then the object follows a periodic trajectory, but that some other initial conditions do not lead to oscillation.

ANSWER: If we let $v = dx/dt$ be the object's velocity, then we may trace the object's movement in phase space as it traces a trajectory $(x(t), v(t))$, starting at the point $(1, 0)$. From the first-order system

$$\frac{dx}{dt} = v$$
$$\frac{dv}{dt} = x(v^2 - 4)$$

we can determine whether this trajectory heads upwards or downwards, to the left or to the right. For example, the trajectory heads roughly southwest from points where $x > 0$ and $v > 2$. Clearly the lines $v = \pm 2$ are trajectories themselves (they correspond to the solutions $x(t) = \pm 2t + x_0$ of the original equation); the origin $x = v = 0$ is the trajectory corresponding to the solution $x(t) = 0$. We can determine the shape of the other trajectories from their slopes dv/dx at each point; with the Chain Rule, the differential equations imply $dv/dx = x(v^2 - 4)/v$. This is a separable differential equation: from $2v\,dv/(v^2 - 4) = 2x\,dx$ we deduce that $\log(|v^2 - 4|) = x^2 - C$, i.e. $v^2 - 4 = Ae^{x^2}$ for some constant A (depending on the trajectory). Note in particular that this allows at most two values of x for each value of v and vice versa; thus if the object should ever return to a position x it has been in twice before, it must surely repeat the very same motions, that is, $x(t) = x(t - T)$ for some time interval T. In simple terms: the trajectories that appear to spiral must in fact form closed loops. This applies to all trajectories that start (and must therefore stay) between the trajectories $v = 2$ and $v = -2$, including ours which starts with $v = 0$.

It is a subtle point is whether or not this T really exists: we must decide whether, in a finite amount of time, the object can actually complete the entire loop around phase space, or whether instead it slows and asymptotically approaches some point on the loop, as $t \to \infty$. (The point would necessarily have to have a v coordinate of 0.) We can compute the amount of time needed to follow a trajectory from one point (x_0, v_0) to another. For simplicity, let us consider the case that both $v_i > 0$. Then as time flows, we will follow the curve $v = \sqrt{4 + Ae^{x^2}}$. Since $v = dx/dt$, we can solve the original differential equation by solving the separable equation $dt = dx/\sqrt{4 + Ae^{x^2}}$. Hence the time needed to follow that portion of the curve is given by

$$t_1 - t_0 = \int_{x_0}^{x_1} \frac{dx}{\left(4 + Ae^{x^2}\right)^{1/2}}$$

The particular trajectory proposed in the question starts at $(x_0, v_0) = (1, 0)$, which requires $A = -4/e$, so the object will make it to the opposite extreme position $x_1 = -1$ in $\int_{-1}^{1} \frac{dx}{(4-4e^{(x^2-1)})^{1/2}}$ units of time. This evaluates numerically to about 1.772 but can be proved to be finite without much trouble: the integral *is* improper at the endpoints, but we can get a series expansion for its behaviour at $x = 1 - \varepsilon$: the exponential is approximately $1 - 2\varepsilon$, and the entire denominator is approximately $\sqrt{8\varepsilon}$. In particular, the integral over a small interval $(1 - \varepsilon, 1]$ is approximately $\sqrt{\varepsilon/2}$ and in particular stays bounded.

4. Find a solution $u(x, t)$ valid for all x in $[0, \pi]$ and all $t \geq 0$ to

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0, t) = u(\pi, t) = 0$ for all $t \geq 0$ and initial conditions $u(x, 0) = 0$ and $\frac{\partial u}{\partial t}(x, 0) = 14\sin(5x)$ for all $x \in [0, \pi]$.

ANSWER: We use Separation of Variables, looking for solutions of the form $u(x, t) = X(x)\,T(t)$ for some functions X and T of one variable each. Such a product solves the differential equation iff $XT'' + 2XT' = 2X''T$; dividing by u then shows $(T'' + 2T')/T = 2X''/X$, which must be a constant k (since the left side shows it is independent of x and the right side shows it is independent of t, too). So the data of the problem stipulate that

$$T'' + 2T' = kT,\ 2X'' = kX,\ X(0) = X(\pi) = 0,\ T(0) = 0,\ X(x)T'(0) = 14\sin(5x)$$

The last equation dictates that X be a multiple of $\sin(5x)$, and hence $k = -50$. Then all the conditions are met except for $T'' + 2T' + 50T = 0$ and $T(0) = 0$. (We will choose the multiplier in X to be $14/T'(0)$.)

The polynomial $r^2+2r+50$ has roots $r = -1 \pm 7i$, so the solutions of the linear equation in T are the linear combinations of $e^{-t}\sin(7t)$ and $e^{-t}\cos(t)$; the initial condition $T(0) = 0$ allows only multiples of the former. Thus $T(t) = Ae^{-t}\sin(7t)$ for some A, giving $T'(0) = 7A$; then $X(x) = (2/A)\sin(5x)$ and finally $u(x,t) = X(x)T(t) = 2e^{-t}\sin(7t)\sin(5x)$.

5. Solve the system

$$x'(t)+2y'(t) = 8x(t)+14y(t) \qquad x'(t)+y'(t) = -7x(t)-13y(t) \qquad x(0) = 13, y(0) = -8$$

ANSWER: We may write the given in formation as $AX'(t) = BX(t)$ where $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and A and B are the matrices $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 14 \\ -7 & -13 \end{pmatrix}$. This is equivalent to the simpler system $X'(t) = C\,X(t)$ where $C = A^{-1}B = \begin{pmatrix} -22 & -40 \\ 15 & 27 \end{pmatrix}$.

Now it is helpful to diagonalize the matrix C. We first find the characteristic polynomial $\det(C - xI)$ which works out to be $x^2 - 5x + 6 = (x-2)(x-3)$, so the eigenvalues are $+2$ and $+3$. To get some eigenvectors, look for the kernels of $C - 2I$ and $C - 3I$; I find them to be the spans of $(5, -3)$ and $(-8, 5)$ respectively, from which I construct the matrix $P = \begin{pmatrix} 5 & -8 \\ -3 & 5 \end{pmatrix}$; general matrix theory predicts that we should have $C = P\,D\,P^{-1}$ where D is the diagonal matrix $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and you may check that this is the case.

Thus our matrix equation $X'(t) = CX(t)$ may be written $X'(t) = PDP^{-1}X$, or $(P^{-1}X)' = D(P^{-1}X)$. So the entries of the column vector $Y(t) = P^{-1}X(t)$ must be multiples of e^{2t} and e^{3t} respectively. This gives us the general solution $X(t) = P\begin{pmatrix} c_2e^{2t} \\ c_3e^{3t} \end{pmatrix}$. Our particular solution has $X(0) = \begin{pmatrix} 13 \\ -8 \end{pmatrix}$ so $Y(0) = P^{-1}\begin{pmatrix} 13 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and thus

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = X(t) = P\begin{pmatrix} e^{2t} \\ -e^{3t} \end{pmatrix} = \begin{pmatrix} 5e^{2t} + 8e^{3t} \\ -3e^{2t} - 5e^{3t} \end{pmatrix}$$