1. Find the 10^{th} derivative of $\dfrac{6}{x^3+x^2-2x}$

ANSWER Try using the technique of Partial Fractions to write this function as a sum of simple parts. The function turns out to equal

$$\frac{2}{x-1}+\frac{1}{x+2}-\frac{3}{x}$$

from which we see that the nth derivative is

$$(-1)^n \cdot n! \cdot \left(2\cdot(x-1)^{-(n-1)}+(x+2)^{-(n-1)}-3\cdot(x+2)^{-(n-1)}\right)$$

2. Sasha Student has prepared poorly for the Calculus test and thinks that for all differentiable functions f and g it is true that

$$\frac{d}{dx}\left(f(x)g(x)\right)=f'(x)\,g'(x)$$

Amazingly, Sasha used this false result on a particular such product and nonetheless obtained the correct derivative of $f(x)g(x)$! Find a pair $\{f(x),g(x)\}$ of non-constant functions for which this is possible. (A few extra points will be awarded for finding additional, substantially different, such pairs.)

ANSWER We are looking for two functions $f(x)$ and $g(x)$ for which

$$f(x)g'(x)+f'(x)g(x)=f'(x)g'(x)$$

Divide both sides by $f'(x)g'(x)$ to see that we require

$$\frac{f(x)}{f'(x)}+\frac{g(x)}{g'(x)}=1.$$

The easiest way to accomplish this is to have each of the summands equal $1/2$, which requires $f'(x)=2f(x)$ and $g'(x)=2g(x)$. This occurs if (and only if) each of the two functions is a multiple of e^{2x}.

Many other solutions are possible: we can have

$$\frac{f(x)}{f'(x)}=\left(\frac{1}{2}\right)(1+h(x)),\qquad \frac{g(x)}{g'(x)}=\left(\frac{1}{2}\right)(1-h(x))$$

for any function h. You may recognize the expression $f'(x)/f(x)$ from "logarithmic differentiation": it's the derivative of $F(x) = \ln(|f(x)|)$. So we need to compute functions F and G which satisfy $F'(x) = 2/(1 + h(x))$ and $G'(x) = 2/(1 - h(x))$, from which we will obtain our answers:

$$f(x) = e^{2 \int dx/(1+h(x))}, \qquad g(x) = e^{2 \int dx/(1-h(x))}$$

There are many kinds of solutions you could try. For example, if $h(x) = kx$ for some constant k then the solutions are

$$f(x) = (1 + kx)^{2/k}, \qquad g(x) = (1 - kx)^{-2/k}$$

e.g. $\{(1 + 2x), 1/(1 - 2x)\}$. Using Partial Fractions you can compute solutions whenever $h(x)$ is any rational function, e.g. for $h(x) = x^2$ we obtain $\{(1 + x)/(1 - x), e^{2\arctan(x)}\}$. And you have the skills to handle certain transcendental functions as well, e.g. when $h(x) = \tan(x)$ we obtain the solution $\{e^x(\cos(x) + \sin(x)), e^x/(\cos(x) - \sin(x))\}$. Other solutions noted by students include $\{x, 1/(1 - x)\}$ and $\{e^x(x - 1), e^{x^2/2}\}$.

3. The equation $x = 2y + 3y^2 + 4y^3$ defines y implicitly as a function of x. (That is, the graph of this equation is the graph of some function $y = f(x)$.) Compute the 0th through 3rd terms of the Taylor series of this function at the origin.

ANSWER We can compute the derivatives of y by implicit differentiation: since $\dfrac{dx}{dy} = 2 + 6y + 12y^2$, we conclude $dy/dx = 1/(2 + 6y + 12y^2)$. Then differentiate using the Chain Rule: $d^2y/dx^2 = -(6+24y)/(2+6y+12y^2)^2 \cdot (dy/dx) = -(6+24y)/(2+6y+12y^2)^3$ and similarly (albeit with more effort) we can compute

$$\frac{d^3y}{dx^3} = \frac{15(24y^2 + 12y + 1)}{8(6y^2 + 3y + 1)^5}$$

Evaluating all these for $x = 0$ (i.e. for $y = 0$) we find

$$f(0) = 0, \qquad f'(0) = \frac{1}{2}, \qquad f''(0) = \frac{-3}{4}, \qquad f'''(0) = \frac{15}{8}$$

so $y = \frac{1}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots$.

It is probably easier to use the method of undetermined coefficients, though: write $y = ax + bx^2 + cx^3 + dx^4 + \ldots$, substitute into the defining equation for y, expand, and collect powers of x. This leads to the equations

$$2a - 1 = 0, \qquad 3a^2 + 2b = 0, \qquad 4a^3 + 6ab + 2c = 0, \qquad 12a^2b + 6ac + 3b^2 + 2d = 0, \qquad \ldots$$

which can be solved successively for the unknown coefficients to obtain

$$a = 1/2, \qquad b = -3/8, \qquad c = 5/16, \qquad d = -15/128, \qquad \ldots$$

4. For what values of x does this series converge?

$$\sum_{n=1}^{\infty} \frac{n^n\, x^{(n^2)}}{n!} = x + 2x^4 + \frac{9}{2}x^9 + \frac{32}{3}x^{16} + \ldots$$

ANSWER The Ratio Test as usually presented for power series does not apply because so many coefficients are zero (that is, $\lim |a_{n+1}/a_n|$ does not exist). But we may still use the Ratio Test for each x, just thinking of this as a series whose n term is as given. That is, the series converges if $L < 1$ where L is

$$\lim_{n\to\infty} \left| \frac{(n+1)^{n+1}\, x^{(n+1)^2}}{(n+1)!} \Big/ \frac{n^n\, x^{(n^2)}}{n!} \right| = \lim_{n\to\infty} \left| \left(1 + \frac{1}{n}\right)^n x^{2n+1} \right| = e \lim_{n\to\infty} \left| x^{2n+1} \right|$$

If $|x| < 1$, the last limit is zero, so $L = 0$ and the series converges. If $|x| > 1$, then $L = \infty$ so the series diverges. If $|x| = 1$ then the limit is $L = e > 1$ so the series diverges again. Thus the series converges for x in $(-1, 1)$.

5. For what values of k does $f(x, y) = \dfrac{x^k y}{x^6 + y^2}$ have a (finite) limit as $(x, y) \to (0, 0)$?

ANSWER Since $ab/(a^2 + b^2) < 1/2$ for all real a, b, we see that for any $k > 3$ we have $f(x, y) < |x|^{k-3}/2 \to 0$. For $k \le 3$ we consider the limit along the curves $y = rx^3$: $f(x, y) = rx^{k-3}/(1 + r^2)$ there, which converges to ∞ if $k < 3$, and which has different limits for different r if $k = 3$. So the limit exists if and only if $k > 3$.

Alternatively, replace x with a different coordinate $u = x^3$; in terms of u and y the question asks for the limit of $(u^{k/3}y)/(u^2 + y^2)$ as $(u, y) \to (0, 0)$. This in turn can be addressed by using polar coordinates in the u, y plane: we need the limit of $r^{(k/3)-1} \cos(\theta) \sin(\theta)$ as $r \to 0$. This limit is zero if $k > 3$, and undefined for $k \le 3$.