BENNETT CALCULUS PRIZE EXAM — ANSWERS Dec 12 2017

1. The equation $y^3 + x^2y + 2x^3 - 3x^2 + 1 = 0$ defines a curve in the plane. We view this curve as the graph of a function $y = f(x)$. Find all the critical points of this function and classify them as local maxima or local minima.

ANSWER: The critical points occur where $dy/dx = 0$. This derivative can be computed by Implicit Differentiation (or using the Implicit Function Theorem); the result is

$$\frac{dy}{dx} = -\left(\frac{\partial F}{\partial y}\right)^{-1}\left(\frac{\partial F}{\partial x}\right) = -\frac{2xy + 6x^2 - 6x}{3y^2 + x^2}$$

This derivative can only be zero when the numerator is, which requires $x = 0$ or $y = 3 - 3x$. In the former case the defining equation requires $y = -1$. In the latter case it requires

$$0 = 27(1-x)^3 + 3x^2(1-x) + (2x^2 - x - 1)(x-1) = (1-x)(28 - 53x + 28x^2)$$

whose only root is $x = 1$. So our only critical points are at $(x, y) = (0, -1)$ and $(1, 0)$.

To determine the local nature of the critical points we will use the second derivative test. We may differentiate the first derivative shown above, using the Chain Rule: $dA(x, y(x))/dx = (\partial A/\partial x) + (\partial A/\partial y) \cdot (dy/dx)$. Since we wish to evaluate this second derivative only at points where dy/dx is already zero, we need only compute $(\partial A/\partial x)$ by the Quotient Rule. The denominator will not affect the sign, and the numerator is $-(3y^2 + x^2)(2y + 12x - 6) + (2xy + 6x^2 - 6x)(2x)$, which evaluates to $+24$ at $(0, -1)$ and to -6 at $(1, 0)$, making the former a local minimum and the latter a local maximum.

You were not expected to do so but there are various additional details you might wish to consider. For example, there are also critical points where dy/dx fails to exist, but the denominator of that derivative is never zero on our curve. The claim that the curve defines a function of x implicitly requires that for each x there be exactly one real y consistent with the initial cubic equation. That is the case here: I have checked the discriminant of the cubic; it's a degree-6 polynomial in x which is everywhere negative.

2. Show that $I = \int_1^2 \frac{1}{4 + x^4}\,dx$ lies between $\frac{1}{20}$ and $\frac{7}{24}$. Five points extra credit goes to the contestant who finds the smallest such interval containing the value of I.

ANSWER: On the interval $[1, 2]$ the denominator of our integrand $f(x)$ is clearly increasing, so $f(x)$ itself is certainly decreasing: $f(1) \geq f(x) \geq f(2) = 1/20$. Since the interval has length 1, we already have $I > 1/20$. In the other direction note that $4 + x^4 > x^4$ so $f(x) < x^{-4}$ and then $I < \int_1^2 x^{-4}\,dx = (-1/3)x^{-3}|_1^2 = 7/24$. So now we know $I \in [0.05, 0.29166]$.

You could obtain many other bounds. For example, one could split the interval $[1, 2]$ in half and on each half obtain tighter bounds using the very same ideas as in the previous paragraph. One could use the inequality $f(x) < 1/(4 + x^2)$ to get an upper bound for I that involves an arctangent. (Numerically, the conclusion is $I < 0.16088$.) Or use the inequality $f(x) > 1/(2 + x^2)^2$ to get a different arctangent bound. (This one numerically is $I > 0.06008$.)

The actual numerical value of the integral is about 0.1159119021. You can obtain this value exactly! Since f is a rational function, we may compute the integral using Partial Franctions. By any of a number of little tricks you might obtain a factorization $4+x^4 = (x^2-2x+2)(x^2+2x+2)$, and each of those factors is quickly seen to be irreducible. Using the usual Partial Fractions techniques we find

$$f(x) = \frac{\frac{1}{8}(x+2)}{x^2+2x+2} + \frac{\frac{1}{8}(-x+2)}{x^2-2x+2}$$

An antiderivative is then

$$\frac{1}{16}\ln(x^2+2x+2) + \frac{1}{8}\arctan(x+1) - \frac{1}{16}\ln(x^2-2x+2) + \frac{1}{8}\arctan(x-1)$$

and the integral is $\frac{1}{8}(\arctan(3) - \arctan(2) + \arctan(1)) = \frac{1}{8}\arctan(4/3)$.

3. Does the following series converge? (Why or why not?)

$$\begin{aligned}&\sum_{k=0}^{\infty}\left(3\cdot\frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5}\right)\\ =&3\cdot\frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + 3\cdot\frac{\ln 6}{6} - \frac{\ln 7}{7} - \frac{\ln 8}{8} - \frac{\ln 9}{9} + 3\cdot\frac{\ln 10}{10} - \cdots\end{aligned}$$

ANSWER: Yes, the series converges, but it is not trivial to demonstrate this. Since not all terms on the right are positive, the integral test and comparison tests do not apply; since the terms do not strictly alternate in sign we cannot use the Alternating Series Test; and we cannot use the Absolute Convergence Test because the series only converges conditionally (which in turn means the Ratio Test and Root Test cannot be conclusive). So we will instead prove convergence by estimating the size of the long expression on the left (the sum of *four* fractions). First, $\log(x)/x$ is a decreasing function for $x > e$ (i.e. where the first derivative is negative) so the first of the summands is larger than each of the other three, making the whole expression positive. This allows us to use a comparison test if we can bound the k-th term suitably.

Since the logarithm is an increasing function, for any $x > 0$ we have

$$3\frac{\ln(x)}{x} - \frac{\ln(x+1)}{x+1} - \frac{\ln(x+2)}{x+2} - \frac{\ln(x+3)}{x+3} < \ln(x)\left(\frac{3}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \frac{1}{x+3}\right)$$

The sum of those four last fractions is

$$\frac{6x^2+22x+18}{x(x+1)(x+2)(x+3)} < \frac{6x^2+24x+18}{x(x+1)(x+2)(x+3)} = \frac{6}{x(x+2)} < \frac{6}{x^2}.$$

We use this inequality for each $x = 4k+2$ to bound our k-th term by $6\ln(4k+2)/(4k+2)^2$. The sum of all these converges, as shown by the integral test or a comparison test to, say, the p-series $\sum 6\sqrt{4k+2}/(4k+2)^2$. Therefore our original series converges too, by the Comparison Test.

One may obtain better estimates of the sizes of the terms by writing each $4k+i$ as $(4k)(1+i/(4k))$ and then using the multiplicative property of logarithms, together with Taylor expansions for $\ln(1+x)$ and for $1/(1+x)$.

Even better, it is possible to actually compute the value of the sum! To do so was a challenge on the 2017 Putnam competition; the sum turns out to be $(\ln(2))^2$.

4. Compute the limit or show that the limit does not exist:

$$\lim_{(x,y)\to(0,0)} (\cos(x+y))^{\cot(x^2-xy+y^2)}$$

ANSWER: The limit does not exist.

First note that for small x and y, $\cos(x+y) \approx 1$ and in particular this value is positive; furthermore $x^2 - xy + y^2$ is never zero unless $x = y = 0$ (the discriminant is negative) so for all small x, y, $\cot(x^2 - xy + y^2)$ is defined. So the function we are to take the limit of is well-defined for all points near, but not equal to, the origin, and may be written as $e^{f(x,y)}$ where $f(x,y) = \cot(x^2 - xy + y^2)\cdot\ln(\cos(x+y))$. If the limit of f along some path is L then the limit of the original function along the same path is e^L.

So we consider the limit along the x-axis, where $y = 0$. There, $f(x,0) = \ln(\cos(x))/\tan(x^2)$. We may compute the limit using L'Hopital's Rule: it's the same as the limit of

$$\frac{-\sin(x)/\cos(x)}{2x\sec^2(x^2)} = \frac{-\cos^2(x^2)}{2\cos(x)}\cdot\frac{\sin(x)}{x} = -\frac{1}{2}$$

But the limit along the line $y = x$ is computed similarly as the limit of $f(x, x) = \ln(\cos(2x))/\tan(x^2)$ and as above L'Hopital's Rule shows this to be equal to the limit of $-(2\sin(2x))/(2x) = -2$. So the limits along two paths are different and therefore there is no (single) limit of our original function as we approach the origin.

Using the Taylor series of the cosine, the logarithm, and the cotangent, we can show more precisely that $f(x, y) = -(x+y)^2/2(x^2 - xy + y^2) + O(r^2)$ where r is the distance from the origin; but that ratio of quadratic polynomials has no limit as $r \to 0$.

5. Compute

$$\int_{y=0}^{1} \left(\int_{x=0}^{1} \frac{x-y}{(x+y)^3} dx \right) dy \qquad \text{and} \qquad \int_{x=0}^{1} \left(\int_{y=0}^{1} \frac{x-y}{(x+y)^3} dy \right) dx$$

ANSWER: Assuming $y \neq 0$, the inner integral on the left may be computed with the Fundamental Theorem of Calculus: since the integrand is $((x+y) - 2y)/(x+y)^3 = (x+y)^{-2} - 2y(x+y)^{-3}$, an antiderivative is $-(x+y)^{-1} + y(x+y)^{-2} = -x(x+y)^{-2}$, so the integral is $-1/(y+1)^2$. (When y *is* equal to 0, the inner integral is improper and in fact does not converge.) Integrating this function now from $y = 0$ to $y = 1$ is not *quite* suitable for the FTC precisely because the integrand is not defined at $y = 0$, but this improper integral will have the same value as $\int_0^1 -(y+1)^{-2}\, dy$ itself; there, an antiderivative is $(y+1)^{-1}$ and so the integral is $-1/2$.

The calculations on the right integral will be nearly identical but off by a minus sign. Equivalently, the change-of-variables transformation $F(x, y) = (y, x)$ changes the second integral into the negative of the first. Either way, the second integral evaluates to $+1/2$.

It is incorrect to claim, using Fubini's Theorem, that the two integrals should be equal, or that either should equal the integral of $(x-y)/(x+y)^2$ over the unit square S, because Fubini's Theorem assumes the integrand is continuous over a rectangle, and this function has a singularity at the origin. Indeed if you want to compute the surface integral $\int\int_S (x-y)/(x+y)^3\, dA$, you should probably use polar coordinates to evaluate the integral on the portion of S that lies a small distance R away from the origin, and then let $R \to 0$. By symmetry, the integrals over the subregions where $y > x$ and then where $y < x$ are negatives of each other, so the total integral is zero for every R, and so the improper integral $\int\int_S (x-y)/(x+y)^3\, dA$ converges to zero.