1. Compute the value $f''(0)$ of the second derivative of f at $x = 0$, where

$$f(x) = \frac{(1+2x)^{1/2}(1+4x)^{1/4}(1+6x)^{1/6}\dots(1+14x)^{1/14}}{(1+3x)^{1/3}(1+5x)^{1/5}(1+7x)^{1/7}\dots(1+15x)^{1/15}}$$

ANSWER: We will first compute the derivative of $F(x) = \ln(f(x))$. Elementary properties of the logarithm allow us to rewrite this function as an alternating sum

$$F(x) = \frac{\ln(1+2x)}{2} - \frac{\ln(1+3x)}{3} + \frac{\ln(1+4x)}{4} - \frac{\ln(1+5x)}{5} + \dots + \frac{\ln(1+14x)}{14} - \frac{\ln(1+15x)}{15}$$

With the Chain Rule we find the derivative of $\frac{\ln(1+nx)}{n}$ to be $1/(1+nx) = (1+nx)^{-1}$; in turn, the derivative of that is $-n/(1+nx)^2$. Thus we have

$$F'(x) = \frac{1}{1+2x} - \frac{1}{1+3x} + \frac{1}{1+4x} - \frac{1}{1+5x} + \dots + \frac{1}{1+14x} - \frac{1}{1+15x}$$
$$F''(x) = -\frac{2}{(1+2x)^2} + \frac{3}{(1+3x)^2} - \frac{4}{(1+4x)^2} + \frac{5}{(1+5x)^2} - \dots - \frac{14}{(1+14x)} + \frac{15}{(1+15x)^2}$$

In particular, $F'(0) = 0$ and $F''(0) = 7$.

These calculations are helpful because $F'(x) = f'(x)/f(x)$, and so we may compute $f'(x) = f(x)\,F'(x)$. Then it follows from the Product Rule that $f''(x) = f(x)\,F''(x) + f'(x)\,F'(x) = f(x)\,F''(x) + f(x)\,(F'(x))^2$. In particular, since $f(0) = 1$ we have $f''(0) = 1 \cdot 7 + 1 \cdot 0^2 = 7$.

2. Evaluate the limit:

$$\lim_{x \to 0^+} \left(\frac{1 + 2^x + 3^x}{3} \right)^{1/x}$$

ANSWER: The expression $f(x)$ inside the parentheses approaches 1 as $x \to 0$. For example $3^x = e^{\ln(3)x}$ will approach $e^0 = 1$ since the exponential function is continuous. Similarly $2^x \to 1$.

However, the limit of an expression of the form $E(x) = f(x)^{g(x)}$, whose base $f(x)$ tends to 1 and whose exponent $g(x)$ increases to $+\infty$, must be treated carefully: this is an “Indeterminate Form”. (You might compare to the case $f(x) = 1 + x$ and $g(x) = 1/x$, whose limit is e.) We may compute instead

$$\ln\left(\lim_{x\to 0^+} E(x)\right) = \lim_{x\to 0^+} \ln(E(x)) = \lim_{x\to 0^+} g(x)\ln(f(x)) = \lim_{x\to 0^+} \frac{\ln(f(x))}{x}$$

which we will do by using L'Hôpital's Rule. The derivative of $\ln(f(x))$ is a fraction $f'(x)/f(x)$ whose denominator we have already noted tends to 1. As for its numerator, we recall that $\frac{d}{dx}a^x = \ln(a)\,a^x$ so $f'(x) = \frac{1}{3}(\ln(2)\,2^x + \ln(3)\,3^x)$, which approaches $\frac{1}{3}(\ln(2)+\ln(3)) = \frac{1}{3}\ln(6)$ as $x \to 0$. So $f'(x)/f(x) \to \frac{1}{3}\ln(6)$ and then by L'Hopital's Rule $\ln(E) \to \ln(6^{1/3})$. Thus our original expression $E(x)$ approaches $\sqrt[3]{6}$ as $x \to 0$.

3. Evaluate the limit:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k^2}{n^3} + \frac{\sqrt{k}}{n^{3/2}}\right)$$

ANSWER: This sum may be written $\sum_{k=1}^{n} \left(\frac{1}{n}\right) f\left(\frac{k}{n}\right)$ where $f(x) = x^2 + \sqrt{x}$. But this is exactly a Riemann sum for the function f on the interval $[0, 1]$, namely the sum that corresponds to a partition of that interval into n equal subintervals (of width $1/n$), with the function f evaluated at the right-hand endpoint of each subinterval. (The kth subinterval has endpoints $x = (k - 1)/n$ and $x = k/n$.) So we are asked to compute the limit of some Riemann sums of this function on that interval, and in the limit, the widths of these subintervals tends to zero. This is the very definition of the (Riemann) integral of the function! Thus our limit is the same as $\int_0^1 f(x)\,dx$, which we evaluate as the sum of $\int_0^1 x^2\,dx = (1/3)$ and $\int_0^1 x^{1/2}\,dx = (2/3)$. Therefore the value of the original limit is 1 .

4. Compute the integral:

$$\int \frac{d\theta}{5 + 2\cos(\theta)}$$

.

ANSWER: There are a few different ways to discover an antiderivative. Two of them use the following preliminary result:

$$\begin{aligned}
\int \frac{d\theta}{a + b\cos^2(\theta)} &= \int \frac{\sec^2(\theta)\, d\theta}{a\sec^2(\theta) + b} \\
&= \int \frac{du}{(a+b) + au^2} && \text{where } u = \tan(x) \\
&= \frac{1}{a+b}\int \frac{dv}{1+v^2} && \text{where } v = \sqrt{\tfrac{a}{a+b}}\, u \\
&= \frac{1}{\sqrt{a(a+b)}}\arctan\left(\sqrt{\frac{a}{a+b}}\tan(\theta)\right)
\end{aligned}$$

Now, the function which you were asked to antidifferentiate did not include the *square* of the cosine. But we can reduce the original question to this one in a couple of ways. If we multiply by the conjugate we get

$$\begin{aligned}
\int \frac{(5 - 2\cos(\theta))\, d\theta}{25 - 4\cos^2(\theta)} &= 5\int \frac{d\theta}{25 - 4\cos^2(\theta)} - 2\int \frac{\cos(\theta)\, d\theta}{21 + 4\sin^2(\theta)} \\
&= \frac{1}{\sqrt{21}}\arctan\left(\frac{5}{\sqrt{21}}\tan(\theta)\right) - \frac{1}{\sqrt{21}}\arctan\left(\frac{2}{\sqrt{21}}\sin(\theta)\right)
\end{aligned}$$

where the first antiderivative is the formula of the previous paragraph, and the second is computed with a substitution $u = \sin(\theta)$.

Alternatively, we may use the trig identity $\cos(\theta) = 2\cos^2(\theta/2) - 1$ to rewrite the original problem as $\int \frac{2d\phi}{3+4\cos^2(\phi)}$ where $\phi = \theta/2$, giving an antiderivative

$$\frac{2}{\sqrt{21}}\arctan\left(\sqrt{\frac{3}{7}}\tan(\theta/2)\right)$$

This is actually identical to the previous antiderivative; using trig identities, each may be reduced to

$$\frac{1}{\sqrt{21}}\arctan\left(\frac{\sqrt{21}\sin(t)}{2 + 5\cos(t)}\right).$$

This second approach is actually quite general: given any rational function of the six trig functions, the substitution $t = \tan(\theta/2)$ works wonders: you can check that

$$\cos(\theta) = \frac{1-t^2}{1+t^2}, \qquad \sin(\theta) = \frac{2t}{1+t^2}, \qquad d\theta = \frac{2\, dt}{1+t^2}$$

so that any integral of such a function will be transformed into an integral of a rational function of t, which can be methodically antidifferentiated using the technique of Partial Fractions.

5. Let $f(x) = 1/(1+x+x^2)$ and let $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$ be the Maclaurin series for f (i.e. the Taylor series of f around the origin). Compute $c_{36} - c_{37} + c_{38}$.

ANSWER: First recall the factorization $x^3 - 1 = (1-x)(x^2+x+1)$. Thus $f(x) = (1-x)(1-x^3)^{-1}$. We may view the second factor as the sum of an infinite geometric series $1 + x^3 + x^6 + x^9 + \ldots$ (which is convergent if $|x| < 1$), and so (for x in that interval) we have $f(x) = (1-x)\sum x^{3k} = \sum x^{3k} - \sum x^{3k+1}$. In other words, $c_n = 1$ if n is a multiple of 3, $c_n = -1$ if n is one more than a multiple of 3, and $c_n = 0$ otherwise. Thus $c_{36} - c_{37} + c_{38} = 1 - (-1) + 0 = 2$.