ALBERT A. BENNETT CALCULUS PRIZE EXAM SOLUTIONS 5/8/10

Name:______________________________ **UT EID:**__________________________

Present Calculus Course:____________ **Instructor:**_____________________________

Permanent Mailing Address:____

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School (Natural Sciences, Engineering, etc.)____________________________________

Show all work in your solutions; turn in your solutions on the sheets provided. (Suggestion: Do preliminary work on scratch paper that you don't turn in; write up final solutions neatly and in order; write your name on all pages turned in.)

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1. *Find the equation of the plane that passes through the points (1,2,2) and* $(-1,1,3)$ *and is parallel to the line* $x = 1 + 2t$*,* $y = 4 - t$*,* $z = 3t$*.*

The normal to this plane must be perpendicular to the vector $(1,2,2) - (-1,1,3) = (2,1,-1)$ and also normal to the direction vector $(2,-1,3)$ of the line, hence must point in the direction of their cross product, which is $(2,-8,-4)$, so the equation of the plane is $2x - 8y - 4z = D$ for some constant D. Plugging in either of the points gives $D = -22$, so our plane is $2x - 8y - 4z = -22$, or $-2x + 8y + 4z = 22$, or $-x + 4y + 2z = 11$.

2. *Let* $f(x) = \sin(x^3)$*. Find the* 99^{th} *derivative of* f *evaluated at* 0*. That is, find* $f^{(99)}(0)$*.*

Since $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, $\sin(x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+3}}{(2k+1)!}$. The coefficient of x^{99} is $1/33!$, so the 99th derivative at $x = 0$ is $99!/33!$.

3. *Find the point on the ellipse* $\frac{x^2}{4} + \frac{y^2}{9} = 1$ *that is farthest from the line* $2x + y = 10$*.*

Since the line has slope -2, the nearest and farthest points must be where the tangent to the ellipse has slope -2. By implicit differentiation, $dy/dx = -\frac{9x}{4y}$. Setting this equal to -2, we get $y = 9x/8$, and plugging into the equation of the ellipse gives $x = \pm 8/5$, $y = \pm 9/5$. The plus signs are for the closest point, and the minus signs are for the farthest point, namely $(-8/5, -9/5)$.

4. *Let C_1 be the solid cylinder in 3-dimensional space consisting of all points whose distance from the x-axis is not greater than 6. Let C_2 be the solid cylinder consisting of all points whose distance from the y-axis is not greater than 6. If V is the intersection of C_1 and C_2, find the volume of V. (Hint: If T is a plane parallel to the xy-plane, what does the intersection of T with V look like?)*

Being within distance 6 of the y and x axes means that $x^2+z^2 \leq 36$ and $y^2+z^2 \leq 36$, so $|x|$ and $|y|$ are both less than or equal to $\sqrt{36-z^2}$. This means that for any $z \in [-6,6]$, the possible values of x and y form a square of side $2\sqrt{36-z^2}$, hence area $144-4z^2$. Integrating this from $z=-6$ to $z=6$ gives 1152.

5. *Let f be a 3^{rd} degree polynomial. That is, $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. Show that there is at least one number x_0 such that $f(x_0) = 0$.*

First note that $f(x)$ is continuous, being a polynomial, and that $\lim_{x\to\pm\infty} \frac{f(x)}{x^3} = \lim_{x\to\pm\infty}(a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3}) = a$.

If $a > 0$ then, when x is sufficiently large and positive, $f(x)$ will be positive, and when x is sufficiently large and negative, $f(x)$ will be negative. Since $f(x)$ is continuous, by the Intermediate Value Theorem, somewhere in between we must have $f(x) = 0$.

If $a < 0$, then for x large and positive we will have $f(x)$ negative, and for x large and negative $f(x)$ will be positive, and we will still have a point in between where $f(x)$ crosses through zero.