1. Compute the integral or explain why the integral does not converge:

$$\int_{-2}^{+1} \frac{(10+4x)}{(5x+x^2)^3}\, dx$$

ANSWER We can calculate the antiderivative using the u-substitution $u = 5x+x^2$: $du = (5+2x)dx$ so

$$\int \frac{(10+4x)}{(5x+x^2)^3}\, dx = \int \frac{2u}{u^3}\, du = -u^{-2} = -(5x+x^2)^{-2}$$

We can then use the Fundamental Theorem of Calculus to compute the integral over any interval on which the integrand is continuous. Sadly, this does *not* include the interval $[-2, +1]$ because of the discontinuity at $x = 0$. But we can compute the integral over $[\epsilon, 1]$ if $\epsilon > 0$; it is $(5\epsilon + \epsilon^2)^{-2} - 1/36$. As ϵ decreases to zero, this half of the integral already diverges, so the original integral diverges too.

REMARK. If you use the method of u-substitution on the *definite* integral, you get $\int_{-6}^{6} \frac{2u}{u^3}\, du$ which is still improper (at $u = 0$) but you might still feel that the answer of 0 (incorrectly obtained from (mis)using the Fundamental Theorem) seems like the right answer because of the symmetry of the graph: the region under the curve on the right half is congruent to the region above the curve on the left. And zero is indeed the “Cauchy Principal Value” of the integral, defined near the point of discontinuity as

$$\lim_{\epsilon \to 0^+} \left(\int_{-1}^{-\epsilon} f(u)du + \int_{\epsilon}^{+1} f(u)du \right)$$

But the CPV is *not* properly computed using u-substitutions since (as in this case) a u-substitution need not preserve the property of “approaching the discontinuity equally from both sides”. Indeed in this case you can already see that the symmetric interval $[-6, +6]$ of u’s corresponds to the asymmetric interval $[-2, +1]$ of x’s. If you DO compute the CPV of the original integral, you’ll find it equals $-\infty$.

2. Compute the sum $\displaystyle\sum_{n=0}^{\infty} \frac{6 \cdot 3^n - 2^{n+3} + 3}{4^n}$ or explain why the series does not converge.

ANSWER This is $\displaystyle\sum_{n=0}^{\infty} 6 \cdot (3/4)^n - 8 \cdot (2/4)^n + 3 \cdot (1/4)^n$, a sum of three geometric series, which sum to

$$6\frac{1}{1-\frac{3}{4}} - 8\frac{1}{1-\frac{1}{2}} + 3\frac{1}{1-\frac{1}{4}} = 24 - 16 + 4 = 12$$

(The first few terms appear to be part of an increasing geometric series but that was just a deliberate distraction!)

3. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{\sin(x)-\sin(y)}$ or explain why the limit does not exist.

ANSWER It is easiest to use some trigonometric identities, e.g. $\sin(x) - \sin(y) = 2\sin(\frac{x-y}{2})\cos(\frac{x+y}{2})$, so that the function may be written as $\frac{z}{\sin(z)} \cdot \frac{1}{\cos(w)}$ where $z = \frac{x-y}{2}$ and $w = \frac{x+y}{2}$, both of which also tend to 0, so that both of the factors tend to 1 and our limit is 1 as well.

Alternatively you may write each sine function in terms of its Taylor series and then divide numerator and denominator by $x - y$ (e.g. $(x^3 - y^3)/(x - y) = x^2 + xy + y^2$) but strictly speaking you must be a bit careful not to interchange the two limiting processes of letting the number of terms increase to $+\infty$ and letting the variables x, y decrease to 0. This can be done by simply estimating the values of the sums in terms of $r = \max(|x|, |y|)$.

4. Let A be an $a \times b \times c$ brick in R^3. Then let B be the set of points in R^3 which are outside of A but whose distance from A is less than 1. What is the volume of B?

ANSWER The set B consists of several parts. There is a rectangular panel one unit away from each of the six sides of the brick (giving a total volume of $2ab+2ac+2bc$), a "quarter-round" (one-fourth of a cylinder) along each of the twelve edges (giving a total volume of $\pi(a+b+c)$), and one eighth of a sphere of radius one at each of the eight vertices (giving a total volume of $(4/3)\pi$). Thus the total volume of B is $2ab + 2ac + 2bc + \pi(a + b + c) + (4/3)\pi$

5. Let C be the curve defined by the equation $y^2 = 2x(x+2)(x+8)$, that is,

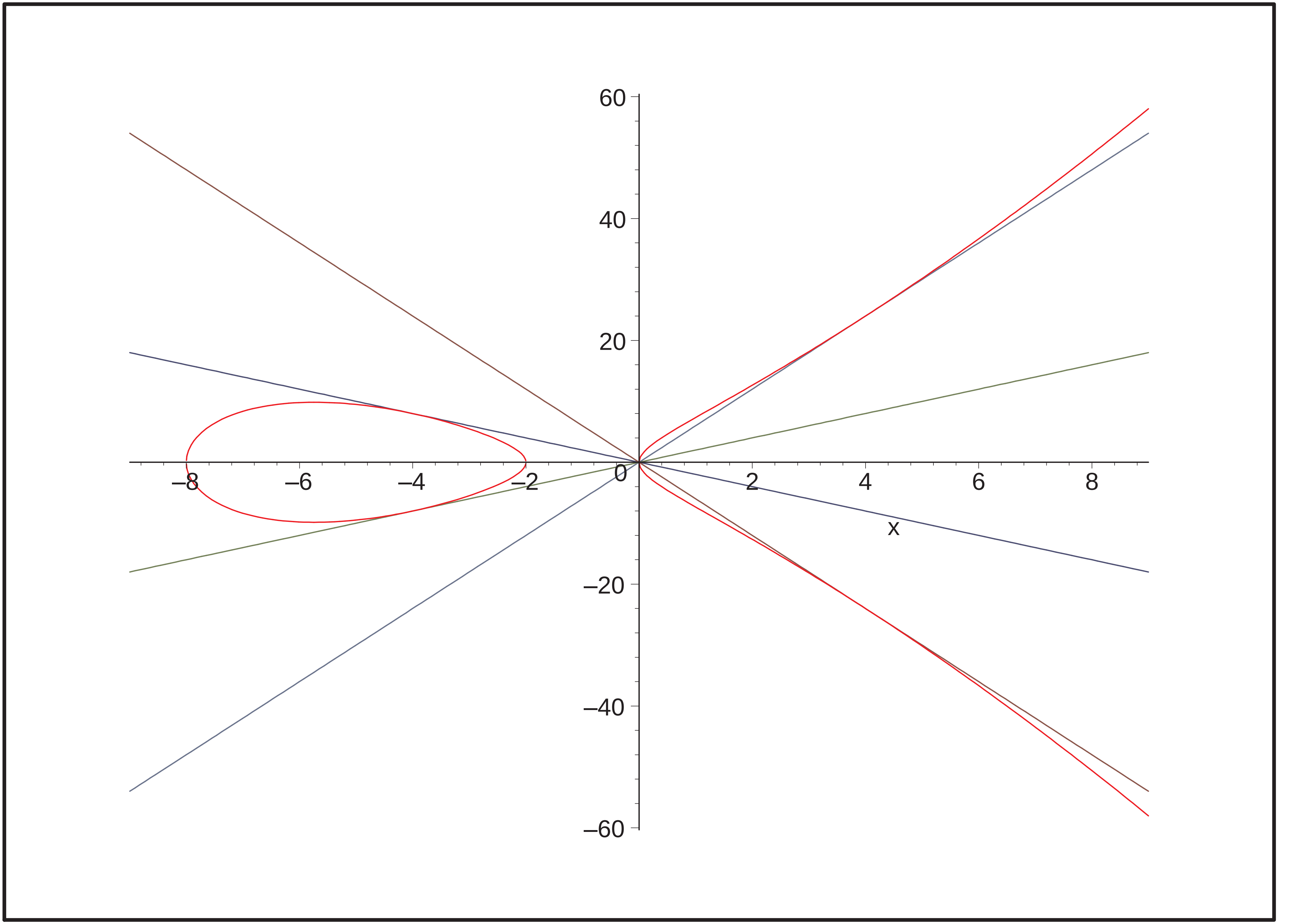
$$C = \big\{(x,y) \,|\, y^2 = 2x(x+2)(x+8)\big\}\,.$$

Find all lines that are tangent to the curve C and which also pass through the origin.

ANSWER Let us consider the line that is tangent to the curve at the point (a,b). The statement that (a,b) is actually on the curve C means that $b^2 = 2a(a+2)(a+8) = 2a^3 + 20a^2 + 32a$. The slope of the tangent line there is the value of dy/dx there, which may be computed by implicit differentiation (or by the Implicit Function Theorem) to be $(6a^2 + 40a + 32)/(2b)$ (and as is suggested by this formula, the tangent line is vertical iff $b = 0$, which happens precisely when $a = 0, -2$, or -8). Of course the slope of the line joining (a,b) to the origin is b/a. So the tangent line and the line joining to the origin are one and the same iff these slopes are equal; that is, the points of interest to us are the ones where $(6a^2 + 40a + 32)/(2b) = b/a$, i.e. where $6a^3 + 40a^2 + 32a = 2b^2$. Since, as remarked earlier, we must also have $b^2 = 2a^3 + 20a^2 + 32a$, this means the only permissible values of a are those with $2a^2 - 32a = 0$, i.e. $a = 0, a = 4$, and $a = -4$. The corresponding values of b are then, respectively, $b = 0$, $b = \pm 24$, and $b = \pm 8$. Thus there are five lines tangent to C which also pass through the origin:

$$x = 0 \qquad y = 6x \qquad y = -6x \qquad y = 2x \qquad y = -2x$$

REMARK. This curve is an example of an "elliptic curve", which has the structure of a mathematical group, and the process of finding these points $P = (a,b)$ where the tangent line passes through $O = (0,0)$ amounts to solving the equation $P + P = O$ in this group.



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