BENNETT CALCULUS PRIZE EXAM — ANSWERS May 8 2017

1. How many real solutions of the equation $e^x + x = 2$ are there? (Hint: first decide how many solutions there are inside the interval $[0, 1]$.)

ANSWER: There is exactly one solution. Let $f(x) = e^x + x$. This function is continuous and takes on the values $f(0) = 1 < 2$ and $f(1) = e + 1 > 2$. Therefore it takes on the value 2 as well, according to the Intermediate Value Theorem. But since $f'(x) = e^x + 1 > 0$ everywhere, it follows that $f(x)$ is increasing everywhere, and hence f can take on any single value at most once.

The solution happens to be approximately $x = 0.4428544010$.

2. If the line $y = mx$ is tangent to the curve $y = e^{ax}$, what is the relationship between m and a?

ANSWER: Let (x_0, y_0) be the point of tangency. Since that point is on the line, we know $y_0 = mx_0$. Since it's also on the other curve, we know $y_0 = e^{ax_0}$. Since the tangent line to the curve there has slope m, we also know $m = ae^{ax_0}$. So there's three equations relating these four values x_0, y_0, m, and a. With some algebra we can solve for three of them in terms of a: $x_0 = 1/a$, $y_0 = e$, and $m = e \cdot a$

For example the curve $y = e^x$ is tangent to the line $y = ex$ at $(x, y) = (1, e)$.

3. The *greatest integer function* is the function whose value at a number x is the largest integer n which is less than or equal to x. This value is denoted by $\lfloor x \rfloor$. For example

$$\lfloor 3 \rfloor = \lfloor \pi \rfloor = \lfloor \frac{7}{2} \rfloor = \lfloor 3.99997 \rfloor = 3 \qquad \text{and} \qquad \lfloor 4 \rfloor = 4$$

Compute $\displaystyle\int_0^\infty \lfloor x \rfloor e^{-x}\, dx$.

ANSWER: Write the integral over $[0, \infty)$ as the sum of the integrals over $[n, n + 1]$ $(n = 0, 1, 2, \ldots)$. On each such interval the integrand is ne^{-x}, the antiderivative is $-ne^{-x}$, and so the integral is $ne^{-n} - ne^{-(n+1)}$. Then the original integral is the sum of these:

$$(0\, e^{-0} - 0\, e^{-1}) + (1\, e^{-1} - 1\, e^{-2}) + (2\, e^{-2} - 2\, e^{-3}) + \ldots = e^{-1} + e^{-2} + e^{-3} + \ldots$$

This is a convergent infinite geometric series which sums to $e^{-1}/(1 - e^{-1}) = 1/(e - 1)$.

(You might want to check that this value is indeed less than $\int_0^\infty xe^{-x}\, dx = 1$.)

4. For what values of x does this series converge?

$$\sum_{n=1}^{\infty} \frac{n!\,x^{2n}}{n^n\,(1+x^{2n})}$$

ANSWER: For every x and n we have $0 < x^{2n}/(1+x^{2n}) < 1$ and so the series is dominated by $\sum n!/n^n$ which converges. (Proof: Its $(n+1)$st term may also be written $a_{n+1} = n!/(n+1)^n$ so the ratio a_{n+1}/a_n is $r_n = (n/(n+1))^n$. The reciprocals of these ratios, $1/r_n = (1+1/n)^n$, converge to e, so $r_n \to 1/e < 1$, and so by the Ratio Test this series converges.) Therefore our series converges for every real x.

The Ratio Test or Root Test may also be applied directly to the original series, but one must consider separately the cases $|x| < 1$, $|x| = 1$, and $|x| > 1$; the limits are a bit troublesome to compute.

5. I have 6 meters of fencing which I wish to use to enclose as much area as I can in two completely completely disjoint regions — one is a square and the other is an equilateral triangle. How much area can I enclose in two such regions having a combined perimeter of 6 meters?

ANSWER: Split the triangle into two right triangles whose short legs have length x. Then the long legs have length $\sqrt{3}\,x$ and the hypotenuses have length $2x$, so that the equilateral triangle has perimeter $6x$ and area $\sqrt{3}\,x^2$. The square is then made of the remaining $6 - 6x$ meters of fencing, so it will have sides of length $\frac{3}{2}(1-x)$ and area $\frac{9}{4}(1-x)^2$. So we consider the total-area function $f(x) = \sqrt{3}\,x^2 + \frac{9}{4}(1-x)^2$ for $x \in [0,1]$. The set of values of f is a compact interval whose minimum and maximum can occur only at $x = 0$, $x = 1$, and where $0 = f'(x) = 2\sqrt{3}\,x - \frac{9}{2}(1-x)$; this last occurs only at $x = x_0 = 9/(9+4\sqrt{3}) = 3(9-4\sqrt{3})/11$ (approximately 0.5650355). But notice that $f''(x) = 2\sqrt{3} + \frac{9}{2} > 0$ so the graph is everywhere concave up (it's a parabola) and so f attains a *minimum* at x_0; the *maximum* must occur at an endpoint. Indeed $f(0) = 9/4 = 2.25$ and $f(1) = \sqrt{3} < 2$, while $f(x_0) = (27\sqrt{3}-36)/11 < 1$ is actually the minimum value of f! (It's about 0.9786702 .) So the maximum value value of f occurs at $x = 0$: the best thing to do is to make only a square.

One may also use Lagrange Multipliers to maximize the area when the sides x of the square and y of the triangle are constrained by $4x + 3y = 6$.

NOTE: the first four questions were taken from the math GRE exam. If you did well on those, perhaps math graduate school is in your future!