BENNETT CALCULUS PRIZE EXAM — ANSWERS May 13 2019

1. For which real numbers r does this limit exist?

$$\lim_{x\to 0^+} x^r \ln(x)$$

ANSWER: For $r<0$ the factors tend to ∞ and $-\infty$ respectively, so the limit diverges to $-\infty$ as well; the case $r=0$ is similar. For $r>0$ write the function as a quotient $\ln(x)/x^{-r}$; this time both numerator and denominator have infinite limits, which means L'Hopital's Rule will apply. That theorem requires us to consider the limit of

$$\frac{1/x}{-rx^{-r-1}} = \frac{1}{-rx^{-r}} = \frac{x^r}{-r}$$

For $r>0$ this limit exists and equals zero, so by the LHopital Theorem, the original limit will also converge to zero.

2. Find an antiderivative of $\cos^4(x) - \sin^4(x)$.

ANSWER: The function factors as $(\cos^2(x)-\sin^2(x))\cdot(\cos^2(x)+\sin^2(x)) = \cos(2x)$, so an antiderivative is clearly $\sin(2x)/2 = \sin(x)\cos(x)$. (One can also evaluate the integral in the form $\int(\cos(x)+\sin(x))(\cos(x)-\sin(x))\,dx$ using Substitution or Integration By Parts, in both cases with $u=\cos(x)+\sin(x)$. Or use the double-angle formulas repeatedly.)

3. Do these series converge or diverge? Explain.

$$(A)\ \sum_{n=1}^{\infty} \sin\left(\frac{\cos(n)}{n^2}\right) \qquad\qquad (B)\ \sum_{n=1}^{\infty} \cos\left(\frac{\sin(n)}{n^2}\right)$$

ANSWER: We know from the graph of the sine function that $|\sin(x)| < |x|$ for all x, so the nth term of series A is smaller in magnitude than $|\cos(n)/n^2| < 1/n^2$ for every n. Since $\sum(1/n^2)$ converges by the Integral Test (it's a "p-series"), we know that series A will also converge by the Comparison Test and Absolute Convergence Test.

Series B diverges, however. For every $n \ge 2$ we have $|\sin(n)/n^2| \le 1/4 < \pi/4$, so (since the cosine function is even and decreasing on $[0,\pi]$) we see $\cos(\sin(n)/n^2) > \cos(\pi/4) = 1/\sqrt{2}$. Hence the Nth partial sum will be larger than $N/\sqrt{2}$, meaning the partial sums diverge to infinity.

Note: series A is neither all-positive nor strictly alternating, so it is more or less inevitable that one must prove *absolute* convergence.

4. Compute $\dfrac{dy}{dx}$ where $y = \arcsin(2uv)$, $u = \cos(x)$, and $v = \sin(x)$. You may assume that $x \in [0, \pi/4]$.

ANSWER: The definition of y implies that $\sin(y) = 2uv = 2\cos(x)\sin(x) = \sin(2x)$. However the range of the arcsin function is $[-\pi/2, \pi/2]$ so y must be the (unique) angle in this interval whose sine matches that of $2x$. As long as $x \in [\pi/4, \pi/4]$, this means $y = 2x$, and as a consequence $dy/dx = 2$.

(Over the whole of $\mathbf{R}$, the graph of $y(x)$ is a sawtooth function with slopes alternating between 2 and -2 on successive intervals of width $\pi/2$. In particular, dy/dx does not exist at $x = \pi/4$ when viewed in this context.)

Of course one may also use the chain rule and the formulas for the derivatives of sine, cosine, and arcsine, but care must be taken with the square roots that arise from the latter

5. A 1-meter-long rod is lying at the base of a 5-meter-tall streetlamp. The rod is oriented north-south. A runner raises the rod to a height of 2 meters and heads east at a rate of 4 meters per second, always keeping the rod perpendicular to his path, level to the ground, and at a height of 2 meters. The rod will then produce a moving shadow on the ground. How rapidly does the width of the rod's shadow increase as the runner moves eastward?

ANSWER: At any moment in the runner's journey we can take two snapshots, one from the side and one from above. From the side we see a right triangle whose legs run up the lamp and from the lamp to the shadow; call their lengths H and L respectively, and let D be the length of the hypotenuse. The horizontal plane containing the rod cuts off a smaller triangle inside this, necessarily similar to the larger triangle, so for some $\varepsilon < 1$ the small triangle has legs εH and εL, and hypotenuse εD.

Now, looking from above we will see an isoceles triangle formed by the lamp and the shadow. The height of this triangle is D and the base has some width w. (It is this dw/dt that we seek.) Again, the plane of the rod cuts off a smaller, similar triangle and the height of this triangle is the hypotenuse of the smaller right triangle in the side view, which had length εD. Thus the length of the rod must, by proportionality, be εw.

But the side view shows us that $H - \varepsilon H$ is the height of the rod above the ground. As long as that height is constant (and the streetlamp's height H is constant!) it follows that ε will remain constant. Of course the width of the rod is also constant, (i.e. εw is constant) and thus w itself is too! So $dw/dt = 0$.

(Indeed you may have noticed that as you walk away from a street lamp your shadow is wider than you at the top but as you walk away the shadow gets longer but no wider.)