Algebraic Topology Final Exam Solutions

1) Let $X$ be a CW complex consisting of one vertex $p$, 2 edges $a$ and $b$, and two 2-cells $f_1$ and $f_2$, where the boundaries of $a$ and $b$ map to $p$, where the boundary of $f_1$ is the loop $ab^2$ (that is, first $a$ and then $b$ twice), and where the boundary of $f_2$ is the loop $ba^2$. Compute the fundamental group of $X$ and the homology groups of $X$.

Back in homework 6 we examined the effect that gluing in a disk has on the fundamental group of a space. If $X$ is obtained from $Y$ by gluing in a disk, then $\pi_1(X)$ is the quotient of $\pi_1(Y)$ by the subgroup generated by the boundary of the disk.

In this case, we first consider the 1-skeleton of $X$, which has fundamental group $\mathbb{F}_2$, with generators $a$ and $b$. We then glue in $f_1$, which means modding out by the cyclic group generated by $ab^2$. We then glue in $f_2$ and mod out by $ba^2$. That is, $\pi_1(X) = \langle a, b \mid ab^2, ba^2 \rangle$. If we use the second relation to set $b = a^{-2}$, we get $\pi_1(X) = \langle a \mid a^{-3} \rangle = \mathbb{Z}_3$.

Now for homology. Using cellular homology, $C_2 = C_1 = \mathbb{Z}^2$, and the boundary map is $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. This is a nonsingular matrix with determinant 3, so the kernel is trivial and the cokernel is $\mathbb{Z}_3$. Thus $H_2 = 0$ and $H_1 = \mathbb{Z}_3$ (and of course $H_0 = \mathbb{Z}$.)

2) Let $G$ be the free group on two generators $a$ and $b$. Show that there exists a finitely-generated subgroup $H$ of $G$ of index 3 that is not normal. Give explicit generators for $H$.

$G$ is the fundamental group of a figure-8, so this is equivalent to finding an irregular triple cover of the figure 8. That’s a chain $\tilde{X}$ with 4 links, which we can picture as a graph with three vertices $p_1$, $p_2$ and $p_3$, and edges $a_1$ from $p_1$ to itself, $b_1$ from $p_1$ to $p_2$, $b_2$ from $p_2$ to $p_1$, $a_2$ from $p_2$ to $p_3$, $a_3$ from $p_3$ to $p_2$, and $b_3$ from $p_3$ to itself.

The fundamental group $\pi_1(\tilde{X}, p_2)$ is a free group on the generators $b_2a_1b_2^{-1}$, $b_2b_1$, $a_2a_3$ and $a_2b_3a_2^{-1}$. This maps to the subgroup $H = \langle bab^{-1}, b^2, a^2, aba^{-1} \rangle$.

Looking at the fundamental group of $\tilde{X}$ based at $p_1$ or $p_3$ instead of $p_2$ would give different index-3 subgroups. One alternate answer is generated by $a$, $b^2$, $ba^2b^{-1}$, and $bab^{-1}b^{-1}$ and equals $bHb^{-1}$, while the other is generated by $b$, $a^2$, $ab^2a^{-1}$ and $abab^{-1}a^{-1}$ and equals $aHa^{-1}$.

3) Let $X$ be a chain with an even number of links (say, viewed as circles of
radius 1 in the \(x-y\) plane, with centers on the \(x\) axis spaced 2 apart) and let \(r : X \to X\) be rotation by 180 degrees about the midpoint of \(X\). Show that any map \(f : X \to X\) that is homotopic to \(r\) has a fixed point.

Viewing \(X\) as a CW complex with \(2n\) links, hence \(2n - 1\) vertices and \(4n - 2\) edges (much like the solution to problem 2), the map \(r\) takes exactly one vertex to itself and doesn’t take any edges to themselves. Thus the trace of \(r\) is 1 on \(C_0\) and zero on \(C_1\), so the Lefschetz number of \(r\) (and the Lefschetz number of any map homotopic to \(r\)) is 1. Since this isn’t zero, any map homotopic to \(r\) has a fixed point.

4) Let \(X_{g,n}\) be the orientable genus-\(g\) surface with \(n\) points removed, where \(n > 0\). Compute the fundamental group and the first homology of \(X_{g,n}\).

We can picture \(X_{g,n}\) as a 4\(g\)-gon with edges identified, with \(n\) points removed from the interior of the 4\(g\)-gon. Let \(U\) be the interior of the polygon, with fundamental group \(F_n\) and generators \(c_1, \ldots, c_n\), where each \(c_i\) is a loop around a hole, connected by a path to our base point. Let \(V\) be a neighborhood of the boundary of the polygon, with fundamental group \(F_{2g}\) and generators \(a_1, \ldots, a_g, b_1, \ldots, b_g\). The intersection is an annulus, so that \(\pi_1(U \cap V) = \mathbb{Z}\), with a generator that goes once around the boundary of the polygon. Viewed in \(U\), this gives \(\prod c_i\). Viewed in \(V\) it gives \(\prod [a_i, b_i]\). Thus we have

\[
\pi_1(X_{g,n}) = \langle \{a_i\}, \{b_i\}, \{c_j\} \mid \prod_i [a_i, b_i](\prod_j c_j)^{-1} \rangle.
\]

We can use the relation to eliminate a single \(c_j\), leaving us with the free group on \(2g + n - 1\) generators (\(g\) a’s, \(g\) b’s and \(n - 1\) surviving c’s.)

To get \(H_1\), we can either abelianize \(\pi_1\) (obtaining \(\mathbb{Z}^{2g+n-1}\)), or we can apply Mayer-Vietoris (most easily with reduced homology) to \(U\) and \(V\). Since \(\tilde{H}_1(U) = \mathbb{Z}^n\) and \(\tilde{H}_1(V) = \mathbb{Z}^{2g}\) and \(\tilde{H}_1(U \cap V) = \mathbb{Z}\) and everything is connected, we have

\[
0 \to H_2(X) \to \mathbb{Z} \xrightarrow{i} \mathbb{Z}^{2g+n} \to H_1(X) \to 0.
\]

All that is left is to identify \(i(1)\), which is the image of the loop around the polygon in both \(U\) and \(V\). In \(U\) it is \(\sum c_j\), and in \(V\) is is \(\sum_i(a_i + b_i - b_i - a_i) = 0\). Thus \(i(1) = \sum c_j\), the kernel of \(i\) is trivial and the cokernel of \(i\) is \(\mathbb{Z}^{2g+n-1}\). Thus \(H_2(X) = 0\) and \(H_1(X) = \mathbb{Z}^{2g+n-1}\).
5) Let $X$ be the 2-sphere with the north and south poles identified. Give a
CW decomposition of $X$ and use this to compute the homology of $X$.

We have a CW composition with one vertex, one edge and one face. Let $p$
be the north pole, which is also the south pole. Let $e$ be the prime meridian,
running from the north pole to the south pole. Let $f$ be the image of a
square, where the $x$ coordinate gives longitude (say, starting at the prime
meridian and running west) and the $y$ coordinate gives latitude (say, with
increasing $y$ meaning going farther south).

The boundary of $e$ is trivial, since the beginning and end points are
identified. The boundary of $f$ is a constant map at the north pole, a path
along $e$, a constant map at the south pole, and a path along $e$ backwards.
Since we traverse $e$ twice, once in each direction, the map $\partial_2$ is zero, so
$H_2 = C_2 = \mathbb{Z}$, $H_1 = C_1 = \mathbb{Z}$ and $H_0 = C_0 = \mathbb{Z}$.

6) Let $\{G_i\}$ be a family of groups, where the index set $I$ is arbitrary. For each
pair $i, j \in I$, let $F_{ij}$ be a (possibly empty) set of homomorphisms $G_i \rightarrow G_j$.
We then define a category as follows:

An object is a group $G$ together with maps $\phi_i : G_i \rightarrow G$ such that, if
$f_{ij} \in F_{ij}$, then $\phi_j \circ f_{ij} = \phi_i$. If $(G, \{\phi_i\})$ and $(G', \{\phi'_i\})$ are two such objects,
then a morphism is a map $\psi : G \rightarrow G'$ such that, for each $i$, $\phi'_i = \psi \circ \phi_i$.

Identify the universal object of this category in the following four circum-
stances. In each case, you should explain your reasoning, but you do not
have to give a complete proof that your answer has the universal property:

(A) When all the families $F_{ij}$ are empty.
(B) When there is a single group $G_1 = \mathbb{Z}$ and a single map $f \in F_{11}$ that is
multiplication by an integer $n$.
(C) When there are three groups $G_{1,2,3}$ and the only nonempty families are
$F_{31}$ and $F_{32}$, each of which consists of a single injection of $G_3$ into $G_1$ or $G_2$.
(D) When the index set is the positive integers each $F_{i,i+1}$ consists of a single
map $f_i : G_i \rightarrow G_{i+1}$, and all other $F_{ij}$'s are empty.

In case (A), when the families are all empty, we have the free product
of the groups $G_i$, since the universal property is precisely the (categorical)
definition of the free product.

In case (B), we have $G = \mathbb{Z}_{n-1}$, since the point 1 is identified with $n$
(unless $n = 1$, in which case we have $\mathbb{Z}$). If $G'$ is any element of our category,
then there is a unique morphism $\mathbb{Z}_{n-1} \rightarrow G'$ sending 1 to $\phi'(1)$. This is
well-defined since \( \phi'(n) = \phi'(1) \), so \( n - 1 \) is in the kernel of \( \phi' \), so the map \( \phi': \mathbb{Z} \to G' \) factors through \( \mathbb{Z}_{n-1} \).

(C) is the usual definition of the amalgamated free product \( G_1 \ast_{G_3} G_2 \).

(D) is the direct limit of the groups \( G_i \), as defined in the last homework.

Because of example (D), some authors (like Serre in his book on trees) call the universal object of our category a direct limit no matter what the groups \( G_i \) or the families \( F_{ij} \) are.