On L'Hôpital's Rule

There are three versions of L'Hôpital's Rule, which I call "baby L'Hôpital's rule", "macho L'Hôpital's rule" and "extended L'Hôpital's rule". The baby and macho versions refer to the problem of evaluating $\lim_{x\to a} f(x)/g(x)$, where $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. In other words, indeterminate forms of the type "0/0", with a finite. (Also to limits as $x \to a^+$ and as $x \to a^-$.) The extended form also applies to forms of the type ∞/∞ and to limits as $x \to \pm\infty$.

1 The three theorems

Theorem 1 (Baby L'Hôpital's Rule) *Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x = a$, with $f(0) = g(0) = 0$. Suppose that f and g are differentiable, and that f' and g' are continuous. Finally, suppose that $g'(a) \neq 0$. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.$$

Also,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

and

$$\lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f'(x)}{g'(x)}.$$

The baby version is easy to prove, and is good enough to compute limits like

$$\lim_{x\to 0} \frac{\sin(2x)}{x + x^2}. \tag{1}$$

However, it isn't good enough to compute limits like

$$\lim_{x\to 0} \frac{1 - \cos(2x)}{x^2}, \tag{2}$$

since in that case $g'(0) = 0$. To solve problems like (2), we need the macho version:

Theorem 2 (Macho L'Hôpital's Rule) *Suppose that f and g are continuous on a closed interval $[a,b]$, and are differentialble on the open interval (a,b). Suppose that $g'(x)$ is never zero on (a,b), and that $\lim_{x\to a^+} f'(x)/g'(x)$ exists, and that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Then*

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Note that this theorem doesn't require anything about $g'(a)$, just about how g' behaves to the right of a. An analogous theorem applies to the limit as $x \to a^-$ (and requires f and g and f' and g' to be defined on an interval that *ends* at a, rather than one that starts at a). You can combine the two to get a theorem about an overall limit as $x \to a$.

The conclusion of Macho L'Hôpital's Rule relates one limit (of f/g) to another limit (of f'/g'), and not to the value of $f'(a)/g'(a)$. This is what allows the theorem to be used recursively to solve problems like (2). Finally, we have the

Theorem 3 (Extended L'Hôpital's Rule) *L'Hôpital's rule applies to indefinite forms of type "∞/∞" as well as "$0/0$", and applies to limits as $x \to \pm\infty$ as well as to limits $x \to a^\pm$. In all of these cases,*

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

2 Proofs of the baby and macho theorems

Suppose that $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then, for any x, $f(x) = f(x) - f(a)$ and $g(x) = g(x) - g(a)$. But then,

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} \\
&= \lim_{x\to a} \frac{[f(x)-f(a)]/(x-a)}{[g(x)-g(a)]/(x-a)} \\
&= \frac{\lim_{x\to a}([f(x)-f(a)]/(x-a))}{\lim_{x\to a}([g(x)-g(a)]/(x-a))} \\
&= \frac{f'(a)}{g'(a)},
\end{aligned}$$

since, by definition, $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and $g'(a) = \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$. Since f' and g' are assumed to be continuous, this is also

$$\frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

That proves the baby version.

To prove the macho version, we first need a lemma:

Theorem 4 (Souped up Mean Value Theorem) *If $f(x)$ and $g(x)$ are continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a point c, between a and b, where*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{3}$$

(When $g(x) = x$, this is the same as the usual MVT.)

Proof of Souped up MVT: Consider the function

$$h(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a)).$$

This is continuous on $[a, b]$ and differentiable on (a, b), with

$$h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)).$$

Note that $h(a) = 0 = h(b)$. By Rolle's Theorem, there a spot c where $h'(c) = 0$. But $h'(c) = 0$ is the same as equation (3).

Proof of Macho L'Hôpital's Rule: By assumption, f and g are differentiable to the right of a, and the limits of f and g as $x \to a^+$ are zero. Define $f(a)$ to be zero, and likewise define $g(a) = 0$. Since these values agree with the limits, f and g are continuous on some half-open interval $[a, b)$ and differentiable on (a, b).

For any $x \in (a, b)$, we have that f and g are differentiable on (a, x) and continuous on $[a, x]$. By the Souped up MVT, there is a point c between a and x such that $f'(c)g(x) = f'(x)g(c)$. In other words, $f'(c)/g'(c) = f(x)/g(x)$. Also, as x approaches a, c also approaches a, since c is somewhere between x and a. But then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(c)}{g'(c)} = \lim_{c\to a^+} \frac{f'(c)}{g'(c)}.$$

That last expression is the same as $\lim_{x\to a^+} f'(x)/g'(x)$.

3 Proving the extended theorem

We're going to use a single trick, over and over again. Namely, we can always rewrite x as $1/(1/x)$, $f(x)$ as $1/(1/f(x))$ and $g(x)$ as $1/(1/g(x))$.

Suppose $L = \lim_{x\to a} \frac{f(x)}{g(x)}$, where both f and g go to ∞ (or $-\infty$) as $x \to a$. Also suppose that L is neither 0 nor infinite. Then

$$L = \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{1/g(x)}{1/f(x)}.$$

Since $1/g(x)$ and $1/f(x)$ go to zero as $x \to a$, we can apply the (baby or macho) L'Hôpital's rule to this limit:

$$\begin{aligned} L &= \lim_{x\to a} \frac{(1/g)'}{(1/f)'} \\ &= \lim_{x\to a} \frac{-g'(x)/g(x)^2}{-f'(x)/f(x)^2} \\ &= \lim_{x\to a} \frac{f(x)^2 g'(x)}{g(x)^2 f'(x)} \\ &= \lim_{x\to a} \frac{f(x)^2}{g(x)^2} \lim_{x\to a} \frac{g'(x)}{f'(x)} \\ &= \frac{L^2}{\lim_{x\to a}[f'(x)/g'(x)]}. \end{aligned}$$

Since $L = L^2/\lim_{x\to a}[f'(x)/g'(x)]$, L must equal $\lim_{x\to a}[f'(x)/g'(x)]$, which is what we wanted to prove.

This argument only works for finite and nonzero values of L. However, if $L = 0$, we can apply the same argument to the limit of $(f(x) + g(x))/g(x)$, which then does not equal zero. The upshot is that

$$1 + \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)+g(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)+g'(x)}{g'(x)} = 1 + \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

hence that $\lim(f/g) = \lim(f'/g')$. Finally, if $\lim(f/g) = \pm\infty$, look instead at $\lim(g/f)$, which is then zero, so the previous reasoning applies. Since $0 = \lim(g/f) = \lim(g'/f')$, $\lim(f'/g')$ must be infinite. By the Souped up MVT, f/g has the same sign as $f'/'g'$, so we must have $\lim(f/g) = \lim(f'/g')$.

Now that we have L'Hôpital's Rule for limits as $x \to a$ (or $x \to a^+$ or $x \to a^-$), we consider what happens as $x \to \infty$. Define a new variable

$t = 1/x$, so that $x \to \infty$ is the same as $t \to 0^+$. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{f(1/t)}{g(1/t)}.$$

But we know how to apply L'Hôpital's Rule to limits as $t \to 0$, so this turns into

$$\lim_{t\to 0^+} \frac{\frac{d}{dt} f(1/t)}{\frac{d}{dt} g(1/t)} = \lim_{t\to 0^+} \frac{-f'(1/t)/t^2}{-g'(1/t)/t^2} = \lim_{t\to 0^+} \frac{f'(1/t)}{g'(1/t)}.$$

Converting back to $x = 1/t$, we get

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)},$$

which is what we wanted. Computing a limit as $x \to -\infty$ is similar, only with $t \to 0^-$ instead of $t \to 0^+$.

That completes the proof of the Extended L'Hôpital's Rule.