Lie Groups Solutions, Problem Set # 1

Section 1.1:

4: This problem can be done either with power series or with differential equations. I'll do part (a) with differential equations and part (b) with power series. For both parts, note that $Xe_1 = x \times e_1 = \|x\|e_2$, $Xe_2 = -\|x\|e_1$ and $Xe_3 = 0$, so the matrix of X in the e basis is $\begin{pmatrix} 0 & -\|x\| & 0 \\ \|x\| & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(a) We will show that $\exp(tX)e_1 = \cos(t\|x\|)e_1 + \sin(t\|x\|)e_2$, $\exp(tX)e_1 = \cos(t\|x\|)e_1 + \sin(t\|x\|)e_2$, and $\exp(tX)e_3 = e_3$. This is true at $t = 0$, since $\exp(0X) = 1$, and the derivative of the right-hand side is X acting on the right-hand side, so by the differential-equations definition of $\exp(tX)$ we are done. Now just set $t = 1$.

(b) From the matrix form, note that $X^3 = -\|x\|^2 X$, and more generally that $X^{2k+1} = (-1)^k \|x\|^{2k} X$ and $X^{2k+2} = (-1)^k \|x\|^{2k} X^2$, so the power series for $\exp(X)$ becomes $1 + X(1 - \|x\|^2/3! + \|x\|^4/5! - \|x\|^6/7! + \cdots) + X^2(1/2 - \|x\|^2/4! + \|x\|^4/6! - \cdots) = 1 + X(\sin\|x\|)/\|x\| + X^2(1 - \cos(\|x\|))/\|x\|^2$.

6: $\exp(\tau D) f(\xi) = \sum_{k=0}^{\infty} \tau^k f^{(k)}(\xi)/k!$. This is precisely the Taylor series for $f(\xi + \tau)$. Since the n-th derivative of f is identically zero, the series terminates after the τ^{n-1} term, and the remainder term in Taylor's theorem is zero, so this is exactly $f(\xi + \tau)$. Notice the interplay between the derivative and the exponential. D can be viewed as an infinitesimal translation. Its exponential is a macroscopic translation (by Taylor's theorem), while the derivative of a translation is (by definition) D.

Section 1.2:

1: If X is nilpotent with $X^k = 0$, then the series for $a = \exp(X)$ terminates after the X^{k-1} term, so there is no issue of convergence. But then $(1 - a)$ is a sum of positive powers of X, so $(1 - a)^k = 0$, so a is unipotent. Likewise, if a is unipotent with $(1 - a)^k = 0$, then the series for $\log(a)$ terminates after $k - 1$ terms, and $X = \log(a)$ is a sum of powers of $(1 - a)$, so $X^k = 0$. This shows that exp maps the nilpotents to the unipotents and that log maps the unipotents to the nilpotents.

What remains is to show that these are inverse operations. This follows from a modification of the Substitution Principle. We already know that after a suitable rearrangement of terms, the power series of $\log(\exp(x))$ is exactly x. But the series for $\log(\exp(X))$ has only a finite number of nonzero terms, so all rearrangements are OK. Likewise for $\exp(\log(a))$.

2: a) If X is semisimple, then $X = PDP^{-1}$, where D is diagonal and the columns of P are the eigenvectors of X. But then $\exp(X) = P\exp(D)P^{-1}$ is also semisimple, with the same eigenvectors and with eigenvalues that are the exponentials of the eigenvalues of X.

b) If a is invertible and semisimple, then $a = PdP^{-1}$, with d diagonal with all nonzero eigenvalues. But then we can take the logs of all of the diagonal entries of d to get a diagonal matrix D with $\exp(D) = d$. Furthermore we can choose our branch for the log function so that the imaginary part of the entries of D are all in $[0, 2\pi)$. But then $\exp(PDP^{-1}) = PdP^{-1} = a$.

c) First note that this is FALSE if we do not make the assumption about eigenvalues of X not differing by multiples of $2\pi i$. The matrices $X = \begin{pmatrix} 0 & 0 \\ 0 & 2\pi i \end{pmatrix}$ and $X' = \begin{pmatrix} 0 & 1 \\ 0 & 2\pi i \end{pmatrix}$ have $\exp(X) = \exp(X') = I$.

Assuming that no two eigenvalues of X differ by a nonzero multiple of $2\pi i$, I claim that a vector v is an eigenvector of $\exp(X)$ if and only it is an eigenvector of X. The "if" follows from part (a). The "only if" depends on the fact that the exponentials of the eigenvalues of X are all different. If v is a nontrivial linear combination of eigenvectors of X with different eigenvalues, then it is a nontrivial linear combination of eigenvectors of $\exp(x)$ with different eigenvalues, and hence is not an eigenvector of $\exp(X)$.

We now proceed to the proposition. If X and X' are simultaneously diagonalizable, with entries differing by multiples of $2\pi i$, then their exponentials are manifestly the same. Conversely, if $\exp(X) = \exp(X')$, then every eigenvector of X' is an eigenvector of $\exp(X') = \exp(X)$, and hence is an eigenvector of X. Thus, X and X' are simultaneously diagonalizable. For the exponentials of the eigenvalues to agree, the eigenvalues must differ by multiples of $2\pi i$.

4: The matrix shown is λ times the unipotent matrix $\begin{pmatrix} 1 & \lambda^{-1} & 0 \cdots & 0 \\ 0 & 1 & \lambda^{-1} \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & 1 \end{pmatrix}$. By problem 1, the unipotent matrix is the exponential of a nilpotent matrix, so the block in question is the exponential of $(\log \lambda)1$ plus a nilpotent matrix.

Now let a be an arbitrary invertible matrix. By choosing the correct basis, we can transform this matrix into the direct sum of Jordan blocks, each of which is an exponential by the previous argument. Since the direct sum of exponentials is the exponential of a direct sum (think about that if it isn't clear!!!), a must be an exponential.