Lie Groups Solutions, Problem Set # 2

Section 1.3:

5: This problem is surprisingly subtle, and took me a while to solve. If $a = \exp(X)$ and $b = \exp(Y)$, showing that $a^{-1} = \exp(-X)$ is in $\exp(\mathbf{n})$ is easy, but showing that $ab \in \exp(\mathbf{n})$ is much harder.

First note that, since $\mathbf{n}$ is finite dimensional and consists of nilpotent matrices, there must be a fixed integer k for which every $X \in \mathbf{n}$ satisfies X^k. By problem 1 of section 1.2 (which you did last week), this implies that $(\exp(X) - 1)^k = 0$ for all $X \in \mathbf{n}$.

Now, for small t, $\exp(tX)\exp(tY) = \exp(Z(t))$, where $Z(t)$ is given by the convergent Dynkin series. Since each term of the Dynkin series is a bracket, and since $[\mathbf{n}, \mathbf{n}] \subset \mathbf{n}$, $Z(t) \in \mathbf{n}$. Hence $\exp(Z(t)) \in \exp(\mathbf{n})$. This implies that $(\exp(tX)\exp(tY) - 1)^k = 0$ for t small.

However, $(\exp(tx)\exp(tY) - 1)^k$ is an analytic function of t, so if it is zero for small t, it must be zero for all t. This implies that the power series for $Z(t) = \log(\exp(tX)\exp(tY))$ has only a finite number of nonzero terms, so it converges for all t, and we conclude that $ab = \exp(Z(1)) \in \exp(\mathbf{n})$.

6: Recall that exp maps the nilpotents to the unipotents bijectively (problem 1.2.1), so for each $a \in \exp(\mathbf{n})$, there is a unique $X \in \mathbf{n}$ for which $a = \exp(X)$.

Now, suppose that $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathbf{n}_1$, that $a = \exp(X)$ and that $b = \exp(Y)$. We have $f(ab) = f(\exp(Z)) = \exp(\phi(Z))$, where Z is given by the Dynkin series. But ϕ of the Dynkin series is the same as the Dynkin series computed from $\phi(X)$ and $\phi(Y)$, since ϕ commutes with brackets. That is, $f(ab) = f(a)f(b)$.

Conversely, if f is a group homomorphism, consider $a(t) = \exp(tX)$ and $b(t) = \exp(tY)$, so $f(a(t)) = \exp(t\phi(X))$, $f(b(t)) = \exp(t\phi(Y))$, and $a(t)b(t) = \exp(tX + tY + t^2[X, Y]/2 + \cdots)$, so $f(a(t)b(t)) = \exp(t\phi(X0 + t\phi(Y) + t^2\phi([X, Y])/2 + \cdots$. Writing the equation $f(a(t)b(t)) = f(a(t))f(b(t))$ as a power series in t and comparing the t^2 coefficients gives $\phi[x, y] = [\phi(X), \phi(Y)]$.

7: (a) There are 1, 2 and 3 dimensional subspaces of the (3-dimensional) set of upper triangular matrices with the desired properties. Notice that the three basis vectors (call them e_α, e_β and e_γ) have all pairwise products equal to zero, except $e_\alpha e_\beta$, which equals e_γ. In particular, the product of any two elements is proportional to e_γ, and the product of any three elements is zero.

ANY 1-D subspace will satisfy $[\mathbf{n}, \mathbf{n}] \subset \mathbf{n}$, since the bracket of a matrix with a multiple of itself is zero. If X is the lone basis element, then $N = \exp(\mathbf{n}) =$

$\{\exp(tX)\} = \{1 + tX + t^2X^2/2\}$. There are two possibilities on what this looks like, depending on whether $X^2 = 0$ or not.

The 2-D subspaces are abelian, and are spanned by e_γ and an arbitrary linear combination of e_α and e_γ. In this case $N = \exp(\mathbf{n}) = 1 + \mathbf{n}$, since for any $X \in \mathbf{n}$ we have $\exp(X) = 1 + X + X^2/2$, and X^2 is proportional to e_γ.

The 3-D subspace is the entire set of upper triangular matrices, for which $N = \exp(\mathbf{n}) = 1 + \mathbf{n}$.

(b) Let $\mathbf{n}$ be the set of real 2×2 traceless matrices. It's easy to see that the bracket of two traceless matrices is traceless (in fact, the bracket of ANY two matrices is traceless, by the cyclic property of traces). Then $\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ are both in $\exp(\mathbf{n})$, but $\begin{pmatrix} -4 & -5 \\ -3 & -4 \end{pmatrix}$ is not, since its trace is less than -2.

Section 2.1:

5: (a) First note that $ij = k = -ji$, so for any complex number α, $j\alpha = \bar{\alpha}j$ and $\alpha j = j\bar{\alpha}$. If $q_1 = \alpha + j\beta$ and $q_2 = \gamma + j\delta$, then $\bar{q}_1 = \bar{\alpha} + \bar{\beta}\bar{j} = \bar{\alpha} - j\beta$, whose matrix is the adjoint of the matrix of q_1. Likewise, $q_1q_2 = \alpha\gamma + j\beta\gamma + \alpha j\delta + j\beta j\delta = (\alpha\gamma - \bar{\beta}\delta) + j(\beta\gamma + \bar{\alpha}\delta)$, whose matrix is $\begin{pmatrix} \alpha\gamma - \bar{\beta}\delta & -\bar{\beta}\bar{\gamma} - \alpha\bar{\delta} \\ \beta\gamma + \bar{\alpha}\delta & \bar{\alpha}\bar{\gamma} - \beta\bar{\delta} \end{pmatrix}$, which is the product of the matrix of q_1 and the matrix of q_2. Since quaternionic multiplication is mapped to multiplication of complex matrices, this gives a homomorphism from $Sp(1)$ (aka the unit quaternions) to the matrices of the given form with $|\alpha|^2 + |\beta|^2 = 1$, and the homomorphism is clearly 1–1. But by example 2 the image is precisely $SU(2)$.

(b) If $\bar{\gamma} = -\gamma$, then the matrix of γ (call it M_γ is anti-Hermitian, so it has pure imaginary eigenvalues and orthogonal eigenvectors. By choosing the phases of the two eigenvectors correctly, we can write $M_\gamma = \lambda PDP^{-1}$, where $P \in SU(2)$, λ is real and positive and $D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Likewise, we can write $M_j = P_0DP_0^{-1}$, so $D = P_0^{-1}M_jP_0$. We then have $M_\gamma = \lambda PP_0^{-1}M_jP_0P^{-1}$. If we take α to be the quaternion whose matrix is $\sqrt{\lambda}P_0P^{-1}$, then $M_\gamma = M_{\bar{\alpha}}M_jM_\alpha$, so $\gamma = \bar{\alpha}j\alpha$.

Note that α is not uniquely defined. Replacing $\alpha' = e^{j\phi}\alpha$ would work as well. This ambiguity corresponds the the phase freedom we have in choosing the eigenvectors of M_γ.

6: This is VERY closely related to problem 7 from section 1.3. (a) $\mathbf{n} = h(3, R)$ is just the upper triangular matrices of problem 1.3.7, and the verification was done there. (b) If $X \in \mathbf{n}$, then $X^3 = 0$, and $\exp(X) = 1 + X + X^2/2$ is in $H(3, R)$. Likewise, if $a \in H(3, R)$, then $(a - 1)^3 = 0$, so $\log(a) = a - 1 - (a - 1)^2/2$, which is easily seen to be in $h(3,R)$. (c) The brackets in $\mathbf{n}$ are: $[e_\alpha, e_\beta] = e_\gamma$, $[e_\alpha, e_\gamma] = [e_\beta, e_\gamma] = 0$. If $Y = \begin{pmatrix} 0 & y_1 & y_3 \\ 0 & 0 & y_2 \\ 0 & 0 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}$, then $Ad(\exp(X))Y = Y + [X,Y] = Y + (y_1 x_2 - y_2 x_1)e_\gamma$, since all higher-order brackets are zero. The adjoint orbit of Y is therefore: (i) Y itself, if $y_1 = y_2 = 0$. In this case Y is proportional to e_γ, and commutes with all elements of the group. (ii) Y plus an arbitrary multiple of e_γ, if $y_1 \neq 0$ or $y_2 \neq 0$.

7: This problem we deleted from the problem set.