Lie Groups Solutions, Problem Set # 3

Section 2.2:

1: (a) If $G$ is the group of invertible block-triangular matrices, then $g$ is the vector space of all block-triangular matrices (with the sizes of the blocks fixed). It is easy to see that the exponential of a block-triangular is block-triangular, and that the derivative of a path in the block-triangulars is block-triangular.

(b) If you add the condition in $G$ that the blocks are identity matrices, then $g$ is the set of upper-block-triangular matrices, i.e. those with $a_k = 0_k \in M_{n_k}$.

3: (a) The condition can be rewritten as $a^t f a = f$. Clearly, if $a \in G$, then $f = (a^{-1})^t a^t f a a^{-1} = (a^{-1})^t f a^{-1}$, so $a^{-1} \in G$. Likewise, if $a$ and $b$ are in $G$, then $(ab)^t f(ab) = b^t a^t f a b = b^t(a^t f a)b = b^t f b = f$, so $ab \in G$. As for the Lie algebra, taking the derivative of $a^t f a = f$ at $a = 1$ gives $X^t f + fX = 0$, so $g = \{X \in M | X^t f = -fX\}$.

(b) If $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $X^t f + fX = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ++ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C - C^t & A^t + D \\ -A - D^t & B - B^t \end{pmatrix}$, so we must have $B$ and $C$ symmetric and $D = -A^t$. That is, the most general Lie algebra element is of the form $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$, with $B$ and $C$ symmetric and $A$ arbitrary.

As for the group, we want $a = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A^t C = C^t A$ (i.e., $A^t C$ symmetric), $B^t D = D^t B$, and $A^t D - C^t B = 1$. For $m = 1$, this is exactly $SL(2, R)$. For $m > 1$, this is more complicated.

4: (a) $so(3)$ is simple. The only sub-algebras are either 1-dimensional (with a trivial bracket), or the full 3-dimensional algebra.

(b) Even though $sl(2, C)$ is the complexification of $so(3)$, the set of available Lie sub-algebras is actually MORE than the complexification of the answer to (a). There exist 2-dimensional subalgebras, all of which are conjugate to the span of $H$ and $X_+$. To see that these are the ONLY 2-dimensional subalgebras, we argue as follows:

Suppose we have a basis for a 2-D subalgebra, spanned by matrices $A$ and $B$. Then $[A, B]$ is a linear combination of $A$ and $B$. By calling this combination our second basis vector and rescaling our vectors, we can assume that $[A, B] = 2B$. If $B$ is semi-simple and has eigenvalues $\pm\lambda$, then $\exp(2\pi B/\lambda) = 1$, so $Ad(\exp(2\pi B/\lambda))A = A$. But by Baker-Campbell-Haussdorff, $Ad(\exp(Bt)A = A + 2Bt$. So $B$ must not be semi-simple, which implies it must be nilpotent, hence conjugate to $X_+$. The equation $[A, B] = 2B$

then implies that $A = H$ plus a multiple of $B$, so our algebra is spanned by $H$ and $X_+$.

8: (a) The group law is trivial, and the Lie algebra is $\{X \in M | Xc = cX\}$, or equivalently $\{X \in M | [X, c] = 0\}$.

(b) If $c$ is diagonal, then it is NOT true that $X$ has to be diagonal. That's only true if the eigenvalues of $c$ are all different. If $c$ has eigenvalue $\lambda_1$ repeated $n_1$ times, then $\lambda_2$ repeated $n_2$ times, etc, then $X$ must be block-diagonal, with the first $n_1 \times n_1$ block arbitrary, the second $n_2 \times n_2$ block arbitrary, etc.

Section 2.3:

9: This is part of Theorem 1 of section 2.6, and a proof can be found on page 78. For completeness, however, I'll reprise the argument here.

Let $g(t) = f(\exp(tX))$. Then $g(0) = 1$ and $g'(t) = d(f(\exp((t+s)X)/dx)|_{s=0} = d(\exp(tX)\exp(sX))/ds|_{s=0} = g(t)\phi(X)$. But the solution to this differential equation is $\exp(t\phi(X))$ satisfies, so $f(\exp(tX)) = \exp(t\phi(X))$. Finally, set $t = 1$.

10: (a) This was essentially done in the proof of Theorem 3. We constructed the analytic map $f(X, Y) = \exp(X)(1 + Y)$ from $M$ to $M$, and noted that by the inverse function theorem it had an analytic local inverse near 1. Since the leaves of $G$ were $V =$ constant (with $V$ denoting the function in the proof, NOT the open ball in $\mathbf{R}^N$), and $G \cup U$ is $C^1$-path-connected, this means that every point in $G \cup U$ has $V = 0$, and hence maps to a ball around zero in $\mathbf{R}^m \times 0 \subset \mathbf{R}^N$.

(b) Part (a) showed that $1 \in G$ has a neighborhood in $G$ which is the restriction of an open set (in $M$) to $G$. Multiplying on the left (or right) by $a$ then gives us a neighborhood of $a \in G$ with the same property. This shows that the intrinsic topology of $G$ is the same as the topology that $G$ inherits from $M$.

(c) First work locally, then glue. Locally, if we have a $C^k$ function on $G$, then it gives a $C^k$ function on $\mathbf{R}^m$, which, when multiplied by a smooth function of the remaining $N - m$ coordinates, gives a $C^k$ function on $\mathbf{R}^N$, hence on a neighborhood of $a$ in $M$. Now glue these local functions together using a smooth partition-of-unity of $M$. This shows that any $C^k$ function on $G$ can be extended to a $C^k$ function on $M$. The fact that the restriction of a $C^k$ function on $M$ to $G$ is $C^k$ is trivial.