Lie Groups Solutions, Problem Set # 8

Section 5.2:

2: In all cases we look at $\tau^{-1}X$, where $\tau \in G$ and $X \in T_\tau G$. For $\mathbf{R}$, $|(\tau, X)| = |(1, \tau^{-1}X)| = |\tau^{-1}||X|$, so the volume form is $d\tau/|\tau|$. For C, the standard volume form is $dxdy = dzd\bar{z}/4i$. The dz terms picks up a factor of τ^{-1}, the $d\bar{z}$ term picks up $\bar{\tau}^{-1}$, and together we get $|\tau|^{-2}$. For the quaternions, there are 4 factors of $d\tau$ in the standard volume form, since $\mathbf{H} \sim \mathbf{R}^4$.

3: (a) Let $X_1, \ldots, X_n$ be the columns of the change in a. The usual volume form on $M_n(\mathbf{R})$ is $d^nX_1 \cdots d^nX_n$. Our left-invariant form is $d^na^{-1}X_1 \cdots d^na^{-1}X_n$. But $d^na^{-1}X_1 = d^nX_1/\det(a)$, so our left-invariant volume form is $da/(\det(a))^n$. Our left-invariant DENSITY is $da/|\det(a)|^n$.

The same argument applies for right-invariant forms and densities, only considering the rows of da instead of the columns.

(b) There is a typo in the book. The formula for $j_l(a)$ applies to $j_r(a)$ and vice-versa.

Before doing a complete calculation, consider the case where a is diagonal. In that case, multiplying a matrix on the left by a^{-1} multiplies the first row by α_1^{-1}, the second row by α_2^{-1}, etc. Since da has n independent entries in the first row, $n-1$ in the second row, etc, the volume form on $a^{-1}da$ is $\alpha_1^{-n}\alpha_2^{1-n}\cdots\alpha_n^{-1}$ times the usual volume form for da.

Likewise, multiplying a matrix on the right by a^{-1} multiplies the first column by α_1^{-1}, the second column by α_2^{-1}, etc. Since there is one entry in the first column of da, two in the second column, etc, the volume form for $da(a^{-1})$ is $\alpha_1^{-1}\cdots\alpha_n^{-n}$ times the usual volume form for da.

In general, a has both diagonal and off-diagonal entries, with the off-diagonal entries all lying above the diagonal. Likewise, a^{-1} has diagonal entries and above-the-diagonal entries. That is, $a^{-1} = A + B$, where A is diagonal with entries α_j^{-1} and B is strictly upper-triangular, hence nilpotent ($B^n = 0$). Left-multiplication by B is likewise nilpotent, so the eigenvalues of "multiply on the left by $A+B$" on da are the same as the eigenvalues of "multiply on the left by A", which we have already computed, so the effect on the volume form for da is exactly as before. Likewise the eigenvalues of "multiply on the right by $A+B$" are the same as those of "multiply on the right by A".

Note that $j_l(a) \neq j_r(a)$, so G is NOT unimodular.

Section 5.3:

2: Looking at the proof of Weyl's integration formula and applying it to $K = SO(3)$, we see that the key is computing the determinant of (Ad t^{-1} -1) acting on a complementary subspace to $\mathbf{t}$. In our case, $\mathbf{g}$ is spanned by the usual generators X_j of rotations about the j axis, and $\mathbf{t}$ is spanned by X_3, and we can take our basis for the complementary subspace to be X_1 and X_2. We compute

$$t^{-1} = t^T = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where c and c are shorthand for the cosine and sine of $2\pi\theta$.

$$t^{-1}X_1 t = \begin{pmatrix} 0 & 0 & -s \\ 0 & 0 & -c \\ s & c & 0 \end{pmatrix} = cX_1 - sX_2.$$

Likewise, $t^{-1}X_2 t = cX_2 + sX_1$, so the eigenvalues of $Ad(t^{-1})$ are $c \pm is$, the eigenvalues of $Ad(t^{-1}) - 1$ are $(c-1) \pm is$, and the determinant of $Ad(t^{-1}) - 1$ is $(c-1)^2 + s^2 = 2 - 2c = 4\sin^2(\pi\theta) = \Delta\bar{\Delta}$, since $\Delta = 2i\sin(\pi\theta)$.

3: (a) Showing that $J = |\det(\mathrm{Ad}(t^{-1}) - 1)|_s$ is word-for-word the same as the proof of Weyl's formula. The key is computing this determinant.

(b) A basis for the Lie algebra of K/T can be viewed as a series of 2×2 blocks, labeled by pairs $i < j$, sitting in the $2i - 1$st and $2i$th rows and $2j - 1$st and $2j$th columns (minus the transpose of this block). $\mathrm{Ad}(t^{-1})$ multiplies this block on the left by $\begin{pmatrix} c_i & s_i \\ -s_i & c_i \end{pmatrix}$ and on the right by $\begin{pmatrix} c_j & -s_j \\ s_j & c_j \end{pmatrix}$, where $s_i = \sin(2\pi\theta_i)$ and $c_i = \cos(2\pi\theta_i)$. That is, if we think of the block as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then multiplying on the left rotates each column by and angle θ_i, and rotates each row by θ_j. The eigenvalues of the left-multiplication are $\exp(\pm 2\pi i\theta_i)$ (each with multiplicity two) the eigenvalues of the right-multiplication are $\exp(\pm 2\pi i\theta_j)$, and the eigenvalues of both actions together are $\exp(\pm 2\pi i(\theta_i \pm \theta_j))$. Subtracting 1 we get that the four eigenvalues of $Ad(t^{-1}) - 1$ for this block are $\exp(\pm 2\pi i(\theta_i \pm \theta_j)) - 1$, and their product is $\Delta_{ij}\bar{\Delta}_{ij}$, where $\Delta_{ij} = (\exp(\pi i(\theta_i + \theta_j)) - \exp(-\pi i(\theta_i + \theta_j)))(\exp(\pi i(\theta_i - \theta_j)) - \exp(-\pi i(\theta_i - \theta_j)))$. Taking the product over all pairs (ij) gives the desired answer.

4: (This problem wasn't assigned, but I'm including it for completeness). The only difference between this and (3) is that there are additional Lie algebra elements coming from the last column (and hence the last row). These can be viewed as a bunch of 2×1 blocks. As in problem 2, each block gets multiplied on the left by a rotation by $2\pi\theta_i$, and get multiplied on the right by 1. This given an additional factor of $(\exp(2\pi i\theta_i) - 1)(\exp(-2\pi i\theta_i) - 1) = |(\exp(\pi i\theta_i) - \exp(-\pi i\theta_i)|^2$.