1) (20 pts) Consider the polar curve \( r = e^\theta \), running from \( \theta = 0 \) to \( \theta = \pi \).

a) Find the slope of a line tangent to this curve at \( \theta = \pi/2 \).

Taking \( \theta = t \), \( r = e^t \) gives \( x = e^t \cos(t) \) and \( y = e^t \sin(t) \), so \( dy/dt = e^t(\sin(t) + \cos(t)) \) and \( dx/dt = e^t(-\sin(t) + \cos(t)) \). At \( t = \pi/2 \) this gives \( dy/dx = e^{\pi/2}/(-e^{\pi/2}) = -1 \).

b) Find the arc-length of this curve.

\[
(dx/dt)^2 + (dy/dt)^2 = 2e^{2t},
\]

so the arclength is \( \int_0^\pi \sqrt{2e^t} \, dt = \sqrt{2}(e^\pi - 1) \).

c) Find the area between this curve and the \( x \)-axis.

\[
\int_0^\pi r^2/2 \, d\theta = \int_0^\pi e^{2\theta}/4 \, d\theta = \left[ \frac{e^{2\theta}}{4} \right]_0^\pi = \left( e^{2\pi} - 1 \right)/4.
\]

2. (20 points) Let \( C \) be a conic section with ellipticity \( e = 1/2 \), focus at the origin, and whose directrix is the line \( x = -2 \).

a) Find the equation of \( C \) in polar coordinates.

Since the distance from the focus to the directrix is \( d = 2 \) and the section opens up to the right, we have

\[
r = \frac{ed}{1 - e \cos(\theta)} = \frac{1}{1 - \cos(\theta)/2}.
\]

This is an ellipse, by the way, since \( e < 1 \).

b) Find an equation of \( C \) in cartesian coordinates.

\[
\begin{align*}
3/4(x^2 - 4x/3) + y^2 &= 1, \\
9/16(x - 2/3)^2 + 3/4y^2 &= 1.
\end{align*}
\]

c) If \( C \) is an ellipse or a hyperbola, find the location of the center and the parameters \( a \) and \( b \). If \( C \) is a parabola, find the location of the vertex.

The center of the ellipse is at \((2/3, 0)\), \( a = 4/3 \) and \( b = \sqrt{4/3} \).
3. (20 pts) Sequences. Evaluate the following limits. Show your work!

a) \( \lim_{n \to \infty} \frac{n^3}{e^n} \)

This is zero, as exponentials grow faster than powers of \( n \). More precisely, by L’Hospital’s rule,

\[
\lim_{n \to \infty} \frac{n^3}{e^n} = \lim_{x \to \infty} \frac{x^3}{e^x} = \lim_{x \to \infty} \frac{3x^2}{e^x} = \lim_{x \to \infty} \frac{6x}{e^x} = \lim_{x \to \infty} \frac{6}{e^x} = 0
\]

b) \( \lim_{n \to \infty} \frac{2n^2 + n + 5e^{-n}}{n^2 + e^{-n}} \)

The key is recognizing that \( n^2 \) grows faster than everything else, while \( e^{-n} \) goes to zero.

c) \( \lim_{n \to \infty} \frac{2n^2 + n + 5e^n}{n^2 + e^n} \)

In this example, \( e^n \) grows faster than everything else.

d) \( \lim_{n \to \infty} \frac{3n^2 + 7 \sin(1/n)}{n} = \lim_{n \to \infty} \frac{3n^2 + 7 \sin(1/n)}{1/n} \)

The first ratio goes to \( \infty \) while the second goes to 1, so the product goes to \( \infty \). That is, the sequence diverges to \( \infty \).
4. (2 pages, 40 points) For each of the following series, indicate whether the series converges absolutely, converges conditionally, or diverges, and why. (E.g. “converges absolutely by the ratio test with $R = 1/2^n$ or “diverges by comparison to $\sum 2^n$” or “converges absolutely by limit comparison to $\sum 1/n^5$”)

a) $\sum_{n=1}^{\infty} \frac{n^2 + n - 7}{n^4 - n^3 + 1}$

This converges by limit comparison to $\sum 1/n^2$. Note that you CANNOT use the basic comparison test, since the ratio is slightly BIGGER than $1/n^2$. Since the terms are already positive, the convergence is absolute.

b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$

This converges by the Alternating Series Test, but its absolute value diverges by (basic) comparison to $\sum 1/n$, and also by the integral test (since $\int_1^\infty \frac{\ln(x)}{x} dx = \int_0^\infty u du$, where $u = \ln(x)$, so the whole thing converges conditionally.

c) $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{10^n}$

This diverges by your choice of (i) the divergence test (since the individual terms do not go to zero), (ii) the ratio test (since $a_{n+1}/a_n = (n+1)/10 \to \infty$), or (iii) the root test with $\rho = \infty$ (with a much uglier computation).

d) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

This converges (absolutely) by the root test, since $(1 - (1/n))^n \to e^{-1} < 1.$

e) $\sum_{n=1}^{\infty} \frac{n^{100} 100^n}{n!}$.

This converges absolutely by the ratio test, with $R = \lim \left(\frac{n+1}{n}\right)^{100} \frac{100}{n+1} = 0.$