Asymptotic analysis of utility-based prices and hedging strategies for utilities defined on the whole real line

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April 6th 2009
Outline

Optimal investment and utility-based pricing hedging

Asymptotic expansions

Summary
The financial model

1. there are $d + 1$ traded (liquid) assets:
   - money market account $B$. We assume the interest rate $r = 0$:
     $B = 1$
   - $d$ stocks: $S = (S^1, \ldots, S^d)$ (semimartingale on the stochastic
     basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$

2. $N$ non-traded or illiquid European contingent claims with:
   - maturity $T$
   - payoff $f = (f_i)_{1 \leq i \leq N}$

Think $N = 1$ for simplicity of notation
The economic agent

1. position \((x, q)\) at time 0:
   - initial capital \(x\), invested in money market and stocks
   - \(q\) units of contingent claims \(f\)

2. time horizon \(T\)

3. preferences over terminal wealth described by a utility function \(U\)
Trading strategies and optimal investment

invests initial (liquid) wealth $x$ holding $H_t$ stocks at any time $t$ ($H$ is predictable and $S$-integrable)

▶ (liquid) wealth process

$$X_t = x + \int_0^t H_u dS_u$$

$\mathcal{X}(x)$ is set of wealth processes with initial capital $x$ (subject to some restrictions depending of the kind of utility)

▶ Total wealth at maturity: $X_T + qf$

Optimal investment with random endowment:

$$u(x, q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + qf)]$$

Denote by $X(x, q)$ the optimal trading strategy above

Remark: when $q = 0$ we have the special case of ”pure investment”

$$u(x) := u(x, 0), \quad X(x) := X(x, q)$$
Utility-based pricing and hedging

Investor with initial position \((x, q)\)

- prices depend on preferences and position \((x, q)\)
- hedging = trading strategy that offsets the risk coming from the contingent claims
- measure risk/return using utility functions
- hedging (of the \(q\) contingent claims) is embedded in the problem of optimal investment with contingent claims: Hodges & Neuberger, Davis, Duffie et al., Henderson, Hobson, etc
Definition of utility-based prices

**Definition 1:** the (vector) \( p = p(x, q) \) is called utility based price for position \((x, q)\) if

\[
u(x, q) \geq u(\tilde{x}, \tilde{q}),
\]

whenever \( x + qp = \tilde{x} + \tilde{q}p \)

**Remark:** pricing by marginal rate of substitution

**Definition 2:** the buyer’s reservation price for \( q \) claims for an agent having \( x \) initial liquid wealth and no claims is defined by

\[
u(x) = u(x - b(x, q), q).
\]
Definition of hedging strategies

**Definition 3:** The number \( c(x, q) \) is called certainty equivalent value of the position \((x, q)\) if

\[
u(c(x, q)) = u(x, q)\]

**Definition 4:** The utility-based hedging strategy for the \( q \) contingent claims is defined by

\[
G(x, q) = X(c(x, q)) - X(x, q)
\]

**Remarks:**

- Split (by definition) the optimal investment strategy into "pure investment" and "hedging"

\[
X(x, q) = X(c(x, q)) - G(x, q)
\]
Approximation of prices and hedging strategies

expansion around \( q = 0 \) (where we can do computations)
  - first order expansion of \( p(x, q) \approx p(x, 0) + D(x)q \) for small \( q \)
  - first order expansion for \( G(x, q) \)
  - second order expansions for \( b(x, q) \) and \( c(x, q) \) for small \( q \)

Two kinds of questions:
  - Quantitative: compute the expansions
  - Qualitative:
    - when is \( D(x) \) symmetric?
    - relate pricing to hedging
    - relate pricing/hedging to quadratic hedging
Answers to previous questions for $U : (0, \infty) \to \mathbb{R}$

- Henderson, Henderson and Hobson: (quantitative) compute second order expansion of reservation prices for $U(x) = \frac{x^{1-p}}{1-p}$, $p > 0$ and basis risk model
- Kallsen: (quantitative and qualitative) first order expansion of utility based-prices for general utility but in the framework of local utility maximization
- Kramkov and S.: (quantitative and qualitative) general utility and general semimartingale model, characterize the qualitative behavior in terms of existence risk-tolerance wealth processes
Objective

- Answer the same questions for a utility function

\[ U : (-\infty, \infty) \rightarrow \mathbb{R} \]

Technical difference:

- \( U : (0, \infty) \rightarrow \mathbb{R} \): easier to define admissible strategies, harder dual problem
- \( U(-\infty, \infty) \rightarrow \mathbb{R} \) harder to define admissible strategies, easier dual problem
Previous work for $U : (-\infty, \infty) \rightarrow \mathbb{R}$

For exponential utility

$$U(x) = -e^{-\gamma x}, \gamma > 0$$

compute expansion of reservation prices and hedging strategies

- Henderson: basis risk model
- Mania and Schweizer, Becherer, Kallsen and Rheinländer, Anthropelos and Zitkovic: more general model (but still has some restrictions), relate to quadratic hedging
Results: mathematical assumptions

Assumptions:
- the stock price process $S$ is locally bounded (or sigma bounded)
- the claim $f$ is bounded (can be relaxed)
- the absolute risk aversion of the utility function is bounded above and below

\[ 0 < c_1 \leq -\frac{U''(x)}{U'(x)} \leq c_2 < \infty. \]

If $V$ is the conjugate of $U$

\[ V(y) = \max_{x \in \mathbb{R}} [U(x) - xy], \ y > 0 \]

then

\[ \mathbb{E}[V(y \frac{dQ}{dP})] < \infty \iff H(Q/P) < \infty \]
More assumptions

Denote
- $\mathcal{M}_a$ the set of absolutely continuous martingale measures,
- $\mathcal{M}_e$ the equivalent martingale measures
- $\mathcal{P}_f$ the measures $\mathcal{Q}$ with finite entropy

$$H(\mathcal{Q}|\mathcal{P}) < \infty$$

Assumption:

$$\mathcal{M}_e \cap \mathcal{P}_f \neq \emptyset$$
Back to optimal investment

Use the framework of Owen-Zitkovic, Schachermayer, six author paper to define admissible strategies as
$$\mathcal{X}(x) = \text{the class of stochastic integrals } X = x + \int HdS \text{ such that } X \text{ is a supermartingale under any absolutely continuous measure } \mathbb{Q} \text{ with finite entropy}$$
$$\mathbb{Q} \in \mathcal{M}_a \cap \mathcal{P}_f$$

- We have a class of admissible strategies which is independent on the utility function, as long as utility satisfies the bounds on the risk aversion
- the optimal investment with random endowment is well posed for any \((x, q)\)
- the indirect utility \(u(x)\) is two-times differentiable and

$$0 < c_1 \leq -\frac{u''(x)}{u'(x)} \leq c_2 < \infty.$$
Asymptotic pricing and hedging: the quantitative question

**Theorem 1:**

- Under previous assumptions, all expansions can be computed, in terms of the second order expansion of the value function $u(x, q)$
- The problem amounts to solving the quadratic optimization problem

$$
\min_{X} \int H dS, H \in \mathcal{H}^2(\mathcal{Q}(y)) \mathbb{E}_{\mathcal{Q}(y)} \left[ \frac{-U''(X_T(x))}{U'(X_T(x))} (X + f)^2 \right]
$$

where $\mathcal{Q}(y)$ is the dual measure

$$
\mathcal{Q}(y) \in \mathcal{M}_e \cap \mathcal{P}_f
$$

(follows from Schachermayer, Owen and Zitkovic)
All questions have positive answer if (and only if) the risk-tolerance wealth process exists

**Definition 5** For fixed $x \in R$, a wealth process $R(x)$ is called risk-tolerance wealth process if

$$R_T(x) = -\frac{U'(X_T(x))}{U''(X_T(x))} > 0$$
Properties of $R(x)$

(in case it exists)

- it is bounded above and below; recall that

\[ 0 < c_1 \leq -\frac{U'(X_T(x))}{U''(X_T(x))} \leq c_2 < \infty \]

- $R_0(x) = -\frac{u'(x)}{u''(x)}$

- it is the derivative of the optimal strategy (when there are no claims):

\[
\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{X(x + \Delta x) - X(x)}{\Delta x}
\]
Existence of $R(x)$

**Theorem 2** For a fixed financial model and utility function the following assertions are equivalent:

- the risk-tolerance wealth process $R(x)$ exists for all $x \in R$
- the dual measure $Q(y)$ does not depend on $y = u'(x)$

**Theorem 3** For a fixed utility function, the following are equivalent:

- the risk-tolerance wealth process is well defined for any financial model
- $U$ is an exponential utility

**Theorem 4** For a fixed financial model, the following are equivalent

- the risk-tolerance wealth process is well defined for any utility function $U : (-\infty, \infty) \to R$
- the set of martingale measures $\mathcal{M}$ admits a largest element $\hat{Q}$ with respect to second order stochastic dominance
Approximation of prices and hedging strategies with risk-tolerance wealth process

Denote

\[ p(x) = p(x, 0) = \mathbb{E}_{Q(y)}[f] \]

The quantity \( p(x) \) is the marginal prices for zero demand (Davis).

**Remark:** the inputs needed to compute \( p(x) \) are obtained solving the “pure investment” problem only:

\[ u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)] \]

The marginal price (at \( q = 0 \)) can be defined as a process

\[ P_t(x) = \mathbb{E}_{Q(y)}[f | \mathcal{F}_t], \ 0 \leq t \leq T \]
Kunita-Watanabe decomposition of the price process

Assume that \( R(x) \) exists, and use it as numéraire:

▶ traded assets \( S^{R(x)} = \left( \frac{R_0(x)}{R(x)}, \frac{R_0(x)S}{R(x)} \right) \)

▶ price process \( \tilde{P}(x) = \frac{R_0(x)P(x)}{R(x)} \)

Adjust the measure \( Q(y) \) to account for the new numéraire

\[
\frac{dQ^{R(x)}}{dQ(y)} = \frac{R_T(x)}{R_0(x)}
\]

Decomposition:

\[ \tilde{P}(x) = \tilde{M} + \tilde{N}, \]

where \( \tilde{M} = p(x) + \int K dS^{R(x)} \), and \( \tilde{N} \) is orthogonal to \( S^{R(x)} \)

**Theorem 5** If there is a risk-tolerance wealth process, then:

▶ \( p(x, q) \approx p(x, 0) + q \frac{u''(x)}{u'(x)} \mathbb{E}_Q(y)[\tilde{N}^2] \)

▶ \( \tilde{G}(x, q) \approx q \tilde{M} \), where \( \tilde{G}(x, q) \) is the hedging strategy measured in units of risk-tolerance
Examples

1. If \( U(x) = -e^{-\gamma x} \) then the risk-tolerance wealth process exists and it is constant

\[
R_t(x) = \frac{1}{\gamma}, \quad 0 \leq t \leq T.
\]

- everything reduces to quadratic hedging under original numéraire and minimal entropy measure.
- recover the results of Mania and Schweizer, Becherer, Kallsen and Rheinländer, Anthropelos and Zitkovic

2. "generalized basis risk model" : \((S, \mathcal{F}^S)\) is complete, general utility function \( U : (-\infty, \infty) \rightarrow \mathbb{R} \)
Extensions

- can relax assumptions of the claims \( f \)
- can consider initial random endowment instead

\[ x \rightarrow g, \]

(as in Anthropelos and Zitkovic)

However, solving the problem for \( x = g \) is as hard as solving the problem for \((x, q)\).
Overview

1. Solve the problem of “pure investment”

\[ u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)] \]

locally around a fixed \( x \in R \). Obtain \( R(x) \) and pricing measure \( \mathbb{Q}(y) \) from here.

2. use \( R(x) \) and \( \mathbb{Q}(y) \) to compute the linear approximation of marginal prices and hedging strategies for all contingent claims \( f \)

Remarks:

- the investment strategy in the presence of claims

\[ u(x, q) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T + qf)] \]

(in the first order) is split into “pure investment” and hedging

- the link between the two operations is provided by \( R(x) \) and \( \mathbb{Q}(y) \)
Summary

- similar results to the case $U : (0, \infty) \to R$ can be proved for $U : (-\infty, \infty) \to R$ under appropriate technical conditions
- pricing and hedging in incomplete markets are parts of investment strategy
- the risk-tolerance wealth process is the natural numéraire for asymptotic pricing and hedging. Utility-based hedging reduces to mean-variance hedging under the new numéraire.
- exponential utility is very peculiar since the risk-tolerance wealth processes are constant