MATH 361K – HOMEWORK ASSIGNMENT 11

Due Tuesday, May 5, 2009

Please write clearly, and staple your work !

1. Problem

Consider the function $f(x) = \tan x = \frac{\sin x}{\cos x}$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(a) Determine $f'(0)$, using the quotient rule for derivatives.

Solution: By the quotient rule,

$$f'(x) = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{(\cos x)^2} = \frac{1}{(\cos x)^2}\,.$$

Therefore, $f'(0) = 1$.

(b) Find a continuous function $\phi(x)$ with the property that $\phi(0) = f'(0)$ (sorry, there was a misprint on the problem sheet) and $f(x) - f(y) = \phi(x)(x - y)$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Solution: Because we want $\phi(0) = f'(0)$, we need $y = 0$. We may choose the function

$$\phi(x) \;=\; \begin{cases} \frac{f(x)-f(0)}{x-0} & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0\,. \end{cases}$$

For $x \neq 0$, this is continuous because $f(x)$ is continuous, and $\frac{1}{x}$ is continuous. In the limit $x \to 0$, we have that

$$\lim_{x\to 0} \phi(x) \;=\; \lim_{x\to 0} \phi(x)\frac{f(x) - f(0)}{x - 0} \;=\; f'(x) \;=\; \phi(0)\,.$$

That is, the limit of function values $\phi(x)$ as $x \to 0$ converges to the function value $\phi(0)$. Therefore, $\phi(x)$ is continuous also at $x = 0$.

2. Problems

Determine the following limits using Bernoulli-de l'Hôpital.

(a) $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$.

Solution: Let $f(x) = \cos x - 1$ and $g(x) = x^2$. Then, clearly, $f(0) = f'(0) = 0$ and $g(0) = g'(0) = 0$, and $f"(0) = -1$ and $g"(0) = 1$. Accordingly,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \;=\; \lim_{x\to 0} \frac{f'(x)}{g'(x)} \;=\; \lim_{x\to 0} \frac{f"(x)}{g"(x)} \;=\; -1\,.$$

(b) $\lim_{x\to 0^+} \frac{\sin x}{x^2+x+x^{10}}$.

Solution: Let $f(x) = \sin x$ and $g(x) = x^2 + x + x^{10}$. Then, clearly, $f(0) = 0$ and $g(0) = 0$, and $f'(0) = 1$ and $g(0) = 1$. Therefore,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} \;=\; \lim_{x\to 0} \frac{f'(x)}{g'(x)} \;=\; 1\,.$$

3. Problems

Use Taylor's theorem with $n = 2$ to approximate e^x at $x = 0$, and give an upper bound on the absolute value of the remainder $R_2(x)$ for $x \in (-1, 1)$.

Solution: Let $f(x) = e^x$. Then, the degree 2 Taylor polynomial at $x = 0$ is given by

$$P_2(x) \;=\; f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 \;=\; 1 + x + \frac{1}{2}x^2$$

and the remainder term by

$$R_2(x) \;=\; \frac{1}{3!}f'''(c)x^3$$

for some $c \in (0, x)$. Since $f'''(c) = e^c$, we find for $c \in (-1, 1)$ that $|f'''(c)| \le e$. Thus, $|R_2(x)| \le \frac{1}{3!}e = \frac{e}{6}$ for $|x| < 1$.

4. Problem

Determine the derivative of $h(x) = \sin(e^{\cos x})$ (use the chain rule twice).

We find

$$h'(x) \;=\; (\cos(e^{\cos x}))e^{\cos x}(-\sin x)\,.$$