PDE I – HOMEWORK ASSIGNMENT 1

Due Friday, September 3, 2010

Please write clearly, and staple your work !

1. Problem

Assume that $u : B_{2r} \subset \mathbb{R}^n \to \mathbb{R}$ satisfies $u\Delta u \geq 0$. Moreover, let $\phi : B_{2r} \to \mathbb{R}$ be non-negative, with $\phi = 0$ on ∂B_{2r}. Prove that then,

$$\int_{B_{2r}} \phi^2 |\nabla u|^2 dx \,\leq\, 4 \int_{B_{2r}} |u|^2 |\nabla \phi|^2 \,.$$

Hint: Start by applying the divergence theorem to $\int_{\partial B_{2r}} \phi^2 u \nabla u \cdot \nu d\sigma = 0$ (why does this hold ?).

2. Problem

As above, assume that $u : B_{2r} \subset \mathbb{R}^n \to \mathbb{R}$ satisfies $u\Delta u \geq 0$. Prove the Cacciopolli inequality

$$\int_{B_r} |\nabla u|^2 \, dx \,\leq\, \frac{4}{r^2} \int_{B_{2r} \setminus B_r} u^2 \, dx \,.$$

Hint: Use Problem 1, with the choice $\phi(x) = 1$ if $|x| \leq r$, and $\phi(x) = \frac{2r - |x|}{r}$ if $r < x \leq 2r$.

3. Problem

Let $u : B_r \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 function with $u = 0$ on ∂B_r. Prove that then,

$$\int_{B_r} u^2 \, dx \,\leq c(n)\, r^2 \int_{B_r} |\nabla u|^2 \,.$$

This is the Dirichlet-Poincaré inequality.

4. Problem

Prove that there exists a constant $C(n)$ such that for all harmonic functions $u : B_{2r} \subset \mathbb{R}^n \to \mathbb{R}$, one has

$$\int_{B_{2r}} u^2 \, dx \,\geq\, (1 + C(n)) \int_{B_r} u^2 \, dx$$

Note that this estimate bounds the rate at which a harmonic function can decay.

Hint: You may want to proceed as follows:

- First, use the Cauchy-Schwarz inequality and problem 2 to prove that

$$\int_{B_{2r}} |\nabla(\phi u)|^2 \, dx \,\leq\, 10 \int_{B_{2r}} u^2 |\nabla \phi|^2 \,.$$

- Next, apply the Dirichlet-Poincaré inequality to the left hand side of this inequality, and choose ϕ as in problem 2.