PDE I – HOMEWORK ASSIGNMENT 6

Due Monday, October 11, 2010. **Please write clearly, and staple your work !**

1. Problem

Prove the following ODE version of the Cauchy-Kovalevskaya theorem using the method of power series and majorants. Assume that $a > 0$, and let $f : (-a, a) \to \mathbb{R}$ be real analytic near $0 \in (-a, a)$. Then, the (unique) solution $u(t)$ of the ODE

$$\dot{u}(t) = f(u(t)) \quad , \quad u(0) = 0 \, ,$$

is also real analytic near 0. You may proceed as follows:

- Show that $f(z) \ll g(z)$ holds, where $g(z) = \frac{Cr}{r-z}$, for $|z| < r$, and some sufficiently small $r > 0$.
- Let $v(t)$ denote the solution of $\dot{v}(t) = g(v(t))$ with $v(0) = 0$. Verify that $v(t) = r(1 - \sqrt{1 - 2Ct/r})$, and show that it is real analytic in a small vicinity of 0.
- $D_t^\alpha u(t)$ can be determined by differentiating the original ODE, for $\alpha \in \mathbb{N}_0$. Conclude by recursion that

$$D_t^\alpha u(0) \; = \; p_\alpha(\{D^\beta f(u(0))\}_{0 \leq \beta \leq \alpha - 1}, \{D_t^\beta u(0)\}_{0 \leq \beta \leq \alpha - 1})$$

  where p_α is a polynomial with non-negative coefficients (determine p_α for $\alpha \leq 3$, and argue by induction why its coefficients are non-negative).
- Argue by monotonicity that $|D_t^\alpha u(0)| \leq D_t^\alpha v(0)$, for all $\alpha \in \mathbb{N}_0$, and conclude the proof.

2. Problem

Consider the second order eigenvalue equation

$$(Lu)(x) \; := \; \frac{1}{2}(-\partial_x^2 + x^2)\, u(x) = \lambda u(x) \, ,$$

for $x \in \mathbb{R}$, $u \in \mathcal{S}(\mathbb{R})$, and $\lambda \in \mathbb{R}$. Define the operators $a := \frac{1}{\sqrt{2}}(\partial_x + x)$ and $a^* := \frac{1}{\sqrt{2}}(-\partial_x + x)$.

- Prove that a and a^* satisfy $(a^* f, g) = (f, ag)$, with respect to the L^2 inner product $(f, g) = \int dx f(x) \overline{g(x)}$, for all $f, g \in \mathcal{S}(\mathbb{R})$.
- Prove that $[a, a^*] = aa^* - a^*a = 1$ (that is, $[a, a^*]f = f$ for all $f \in \mathcal{S}(\mathbb{R})$).
- Prove that $a^*af = (L - \frac{1}{2})f$, and conclude that $(f, Lf) \geq \frac{1}{2}(f, f)$ for all $f \in \mathcal{S}(\mathbb{R})$.
- Determine the solution u_0 of $au_0 = 0$, and conclude that it satisfies $Lu_0 = \frac{1}{2}u_0$.
- Prove that whenever u is an eigenvector of L satisfying $Lu = \lambda u$, it follows that a^*u is an eigenvector for the eigenvalue $\lambda + 1$ (for this purpose, use the commutation relation above to show that $a^*a(a^*u) = a^*(a^*a + 1)u$).
- Conclude that $u_j := (a^*)^j u_0$ satisfies $Lu_j = (\frac{1}{2} + j)u_j$, for $j \in \mathbb{N}_0$.
- Verify that $u_j \in \mathcal{S}(\mathbb{R})$, for every $j \in \mathbb{N}_0$.