PDE I – HOMEWORK ASSIGNMENT 7

Due Monday, October 25, 2010. **Please write clearly, and staple your work !**

1. Problem

Show that an inequality of the form

$$\|\widehat{f}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \quad ,$$

for all $f \in L^q(\mathbb{R}^n)$ can only be true if $\frac{1}{q} + \frac{1}{p} = 1$. Prove that it is satisfied for $1 \leq p \leq 2$. Moreover, show that it does not hold true for $p > 2$.

Hint: Consider scaling for the first part, and interpolation for the second part. For the third part, consider first $n = 1$, and the function $f(x) = \phi(x)e^{-i\lambda x^2}$, for some large $\lambda \gg 1$, where $\phi \in C_0^\infty(\mathbb{R})$ is fixed.

2. Problem

Show that $\|f * g\|^2_{L^2(\mathbb{R})} \leq \|f * f\|_{L^2(\mathbb{R})}\|g * g\|_{L^2(\mathbb{R})}$ for all $f, g \in L^2(\mathbb{R})$. Can there be such an inequality with $L^1(\mathbb{R})$ instead of $L^2(\mathbb{R})$?

3. Problem

Let $f(x, y)$ be a measurable function on $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Prove the Minkowski integral inequality

$$\Big\| \int_{\Omega_2} f(x, y)dy\Big\|_{L^p_x(\Omega_1)} \leq \int_{\Omega_2} \|f(x, y)\|_{L^p_x(\Omega_1)}dy$$

for $1 \leq p \leq \infty$.

Hint: Let $H(x) := \int_{\Omega_2} f(x, y)dy$, and express the left hand side of the above inequality as an integral of the form $(\int_{\Omega_1} g(x)(H(x))^{p-1}dx)^{1/p}$. Then, apply the Holder inequality.

4. Problem

Assume that $\widehat{f}$ has support in the shell $S := \{\xi \in \mathbb{R}^n \,|\, 2^k \leq |\xi| \leq 2^{k+1}\}$. Prove that then,

$$\|D_x f\|_{L^p(\mathbb{R}^n)} \sim 2^k \, \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 \leq p \leq \infty$.

Hint: Write $f = f * \phi_S^\vee$, where ϕ_S is a smooth approximate characteristic function for S. At some point in the proof, Minkowski's integral inequality will be of use.