Exponential functions.

With integer exponents.

$a^1 = a$ $\qquad\qquad a > 0.$

$a^2 = a \cdot a$

$a^3 = a \cdot a \cdot a$

$\vdots$

$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}.$ $\qquad$ "a to the power n".

a base

n exponent.

$$\boxed{a^n \cdot a^m} = \underbrace{a \cdot a \cdots a}_{n} \cdot \underbrace{a \cdot a \cdots a}_{m} = \boxed{a^{n+m}}$$

$n > m$

$$\frac{a^n}{a^m} = \frac{\cancel{\underbrace{a \cdot a \cdots a}_{m}} \cdot \underbrace{a \cdots a}_{n-m}}{\cancel{\underbrace{a \cdot a \cdots a}_{m}}} = a^{n-m} = a^n a^{-m}$$

$$\Rightarrow \boxed{a^{-m} = \frac{1}{a^m}}$$

$$a^0 = a^{1-1} = a^1 \cdot a^{-1} = \frac{a}{a} = 1 \quad \text{for } a > 0$$

(non-zero).

$$0^n = 0 \qquad n > 0.$$

Question: What about 0^0? Is not well-defined.
Can't divide by 0!

$$\boxed{(a^n)^m} = \underbrace{a^n \cdots a^n}_{m \text{ times}.} = \left.\begin{matrix} \underbrace{a \cdots a}_{n} \\ \cdot \underbrace{a \cdots a}_{n} \\ \vdots \\ \cdot \underbrace{a \cdots a}_{n} \end{matrix}\right\} m \text{ times}. \quad = \boxed{a^{nm} = (a^m)^n}$$

Exponentials with fractional powers.

$$\left(a^{5/2}\right)^2 = a^{\frac{5}{\cancel{2}}\cdot\cancel{2}} = a^5 \qquad a > 0.$$

$$a^{5/2} = \sqrt[2]{a^5}$$

$p, q > 0$ integers.

$$\left(a^{p/q}\right)^q = a^{\frac{p}{\cancel{q}}\cdot\cancel{q}} = a^p.$$

$$\Rightarrow a^{p/q} = \sqrt[q]{a^p}, \qquad \boxed{a^{1/q} = \sqrt[q]{a}.}$$

Ex: $2^{1.212} = \sqrt[1000]{2^{1212}} = \sqrt[250]{2^{303}}$

$$\frac{1212}{1000} = \frac{606}{500} = \frac{303}{250}$$

What if exponent is not a fraction?

Ex: a^π $\qquad \pi = 3.1415\ldots\ldots$

a^π close to a^3

closer to $a^{3.1}$

even closer to $a^{3.1415}$

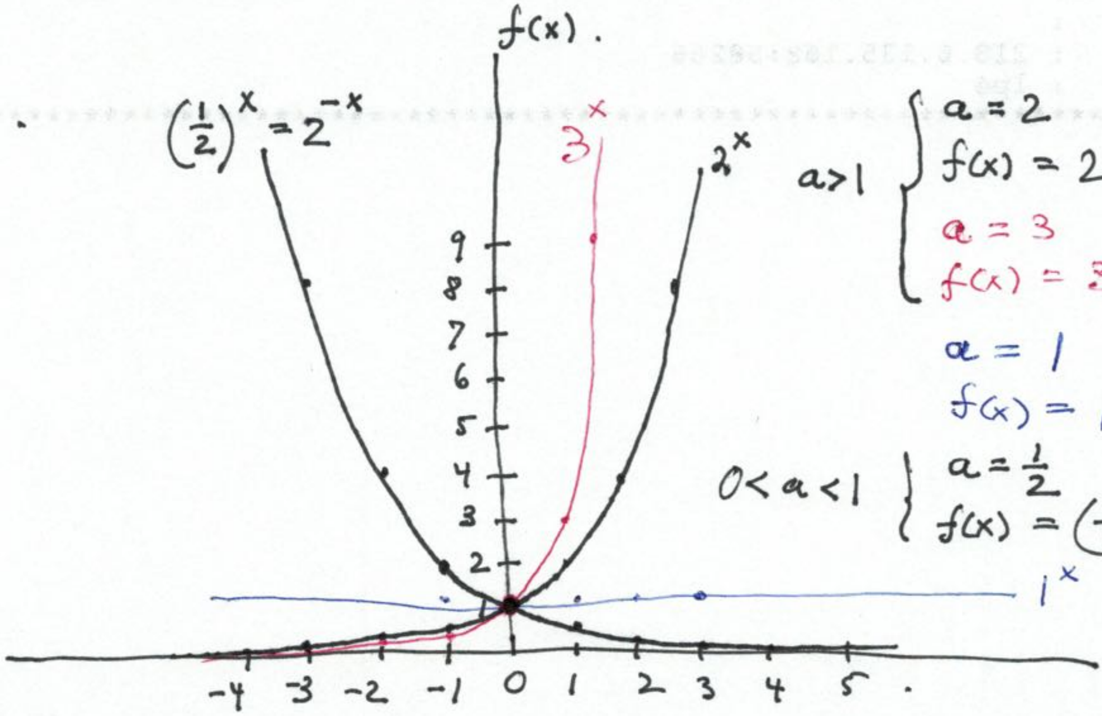
$\vdots$

more and more precise approximations.

$\Rightarrow$ take a limit.

The function $f(x) = a^x$, $a > 0$.

<u>Ex</u>.



$a>1$ $\begin{cases} a = 2 \\ f(x) = 2^x \\ a = 3 \\ f(x) = 3^x \end{cases}$

$a = 1$
$f(x) = 1^x = 1$

$0 < a < 1$ $\begin{cases} a = \frac{1}{2} \\ f(x) = \left(\frac{1}{2}\right)^x \end{cases}$

$$\left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = \frac{1}{2^x}$$

<u>Ex</u>. Growth of colony of bacteria.

Assume that every second, each bacterium splits into 2.

$\Rightarrow$ t time in seconds.

$n(t)$ number of bacteria at time t.

$n(0)$

$n(1) = 2\,n(0)$

$n(2) = 2 \cdot 2n(0) = 2^2 \cdot n(0)$.

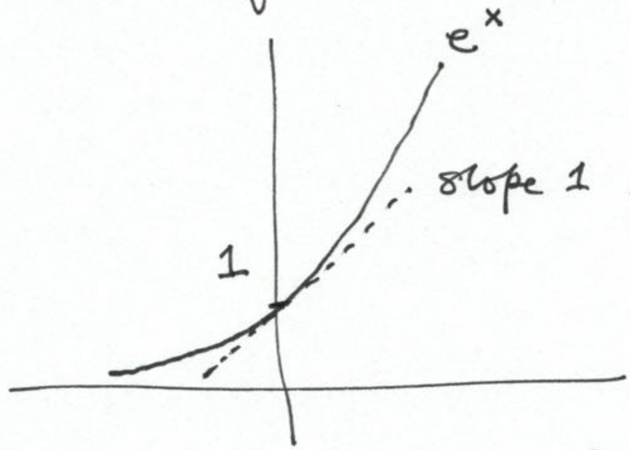
$n(3) = 2 \cdot 2^2\, n(0) = 2^3 \cdot n(0)$.

$\vdots$

$n(t) = 2^t\, n(0)$

There is a unique number for a such that

a^x crosses the y-axis with slope 1 .



$e = 2.71828\ldots.$

Euler number .