Comparison methods.

Thm If $f(x) \le g(x)$ for all x near a (except possibly at a) and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist (not necessarily finite)

Then:

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

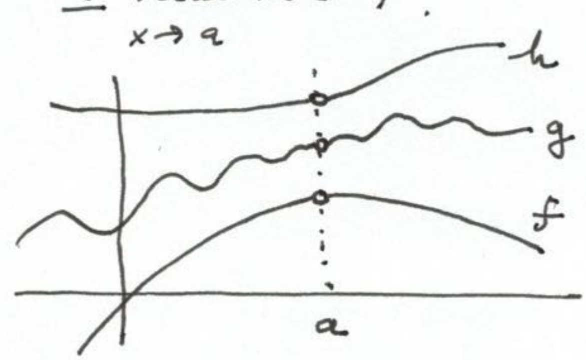
This also holds for $\lim_{x\to a^+}$ or $\lim_{x\to a^-}$ instead of $\lim_{x\to a}$.

Ex $$\lim_{x\to 0^+} \frac{1+x^4}{x^3} \geq \lim_{x\to 0^+} \frac{1}{x^3} = \infty \quad \left(1+x^4 \geq 1\right)$$

Thm (squeeze).

If $f(x) \le g(x) \le h(x)$ for x near a (except possibly at a) and all have well-defined limits (resp. left/right limits) (not necessarily finite).

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to a} h(x).$$

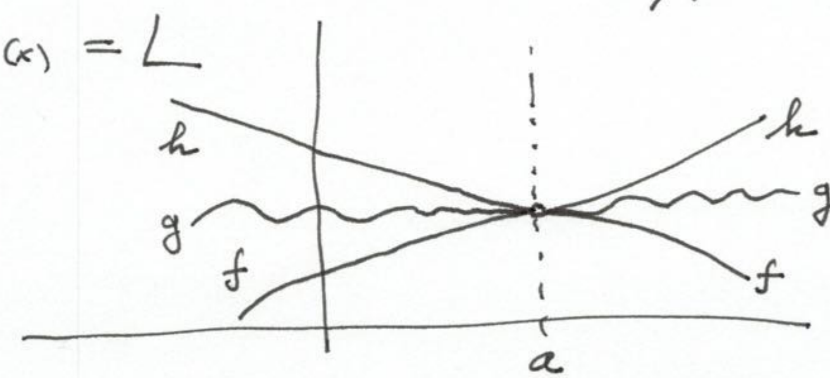


In particular, if

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then

$$\lim_{x\to a} g(x) = L$$



Ex $f(x) = x^2 \sin \frac{1}{x}$.

Find $\lim\limits_{x \to 0} x^2 \sin \frac{1}{x}$.

$\sin \frac{1}{x} \geq -1$ $\quad\quad\quad$ $\sin \frac{1}{x} \leq 1$

$$x^2 \cdot (-1) \leq x^2 \sin \frac{1}{x} \leq x^2 \cdot 1$$

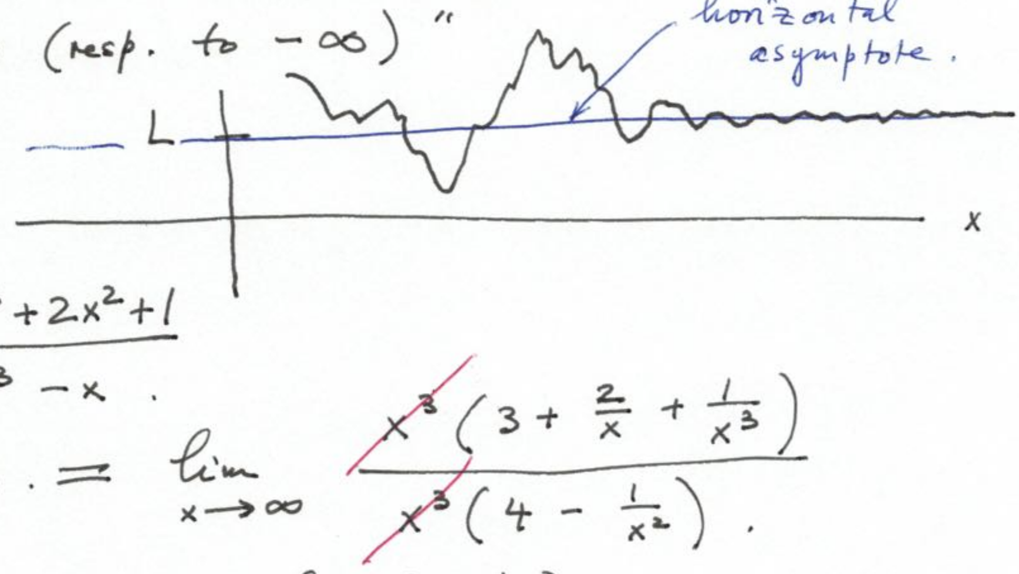
$$\underbrace{\lim_{x \to 0} (-x^2)}_{0} \leq \lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) \leq \underbrace{\lim_{x \to 0} x^2}_{0}$$

$$\Rightarrow \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0.$$

Limits at infinity, horizontal asymptotes.

Def $\lim\limits_{x \to \pm\infty} f(x) = L$

"$f(x)$ approaches L with arbitrary precision as x grows to ∞ (resp. to $-\infty$)"



Ex $f(x) = \dfrac{3x^3 + 2x^2 + 1}{4x^3 - x}$.

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{\cancel{x^3}\left(3 + \frac{2}{x} + \frac{1}{x^3}\right)}{\cancel{x^3}\left(4 - \frac{1}{x^2}\right)}.$$

Limit laws apply when $x \to a$ with $a = \pm\infty$, but $\lim\limits_{x \to \pm\infty} f(x)$ and $\lim\limits_{x \to \pm\infty} g(x)$ have to be finite.

$$= \frac{\lim\limits_{x \to \infty} \left(3 + \frac{2}{x} + \frac{1}{x^3}\right)}{\lim\limits_{x \to \infty} \left(4 - \frac{1}{x^2}\right)} = \frac{3}{4}$$

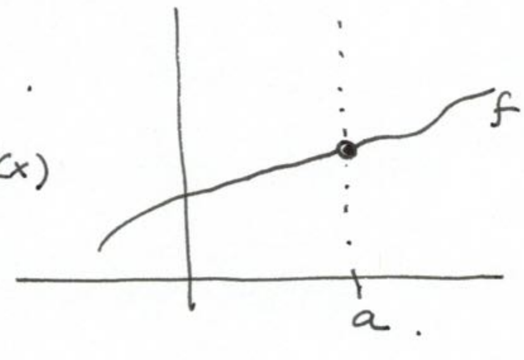
finite limits, limit laws ok.

Continuity of functions.

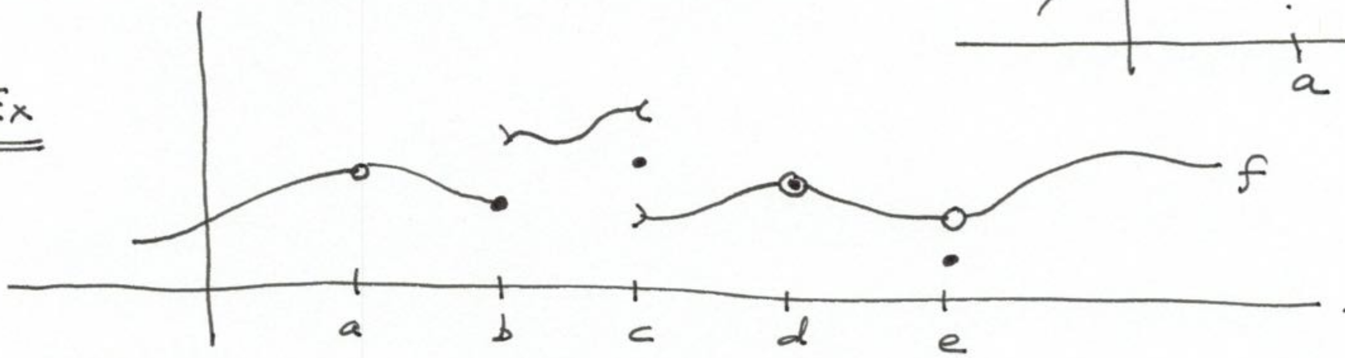
Def f is continuous at $x=a$ if:

1) f has a fct value $f(a)$ at $x=a$.
2) $\lim\limits_{x \to a} f(x)$ is well-defined.
3) Function value = limit.

$$f(a) = \lim_{x \to a} f(x)$$



Ex



	Left limit	Right limit	Limit	Fct value	Continuous
$x=a$:	Y	Y	Y	N	N
$x=b$:	Y	Y	N	Y	N.
$x=c$:	Y	Y	N	Y	N.
$x=d$:	Y	Y	Y	Y	Y.
$x=e$:	Y	Y	Y	Y	N.

Ex: $f(x) = \dfrac{1}{x^2}$

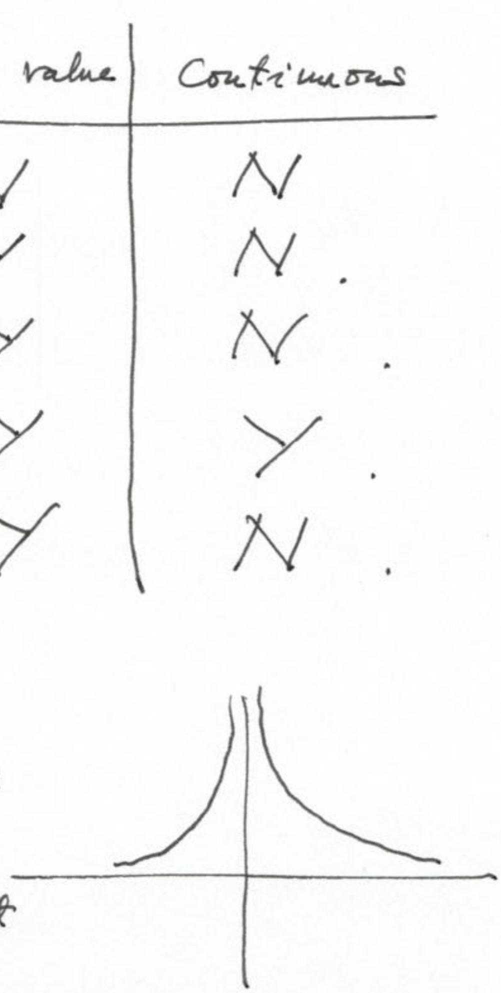
$\lim\limits_{x \to 0^+} f(x) = \infty$

$\lim\limits_{x \to 0^-} f(x) = \infty$

$\lim\limits_{x \to 0} f(x) = \infty$

f has no fct value at $x=0$.

<u>not</u> continuous at $x=0$.



Ex $f(x) = x^2 \sin \frac{1}{x}$.

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

but: f has no fct value at $x = 0$.

$\Longrightarrow$ not continuous at $x = 0$.

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\underset{\text{fct value}}{g(0) = 0} = \underset{\text{limit}}{\lim_{x \to 0} g(x)}. \Longrightarrow g \text{ is continuous at } x = 0.$$