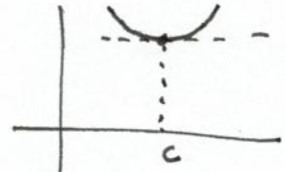
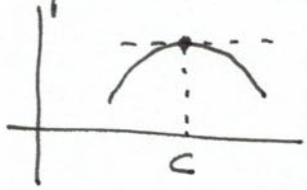
Thm (2nd derivative test) Assume f'' is continuous near c.

1) if $f'(c) = 0$, and $f''(c) > 0$
then f has a local min at c

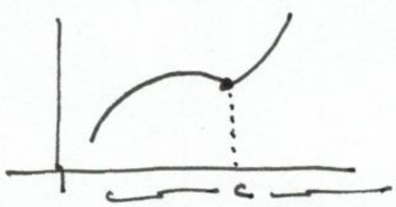


2) if $f'(c) = 0$, and $f''(c) < 0$
then f has a local max at c.



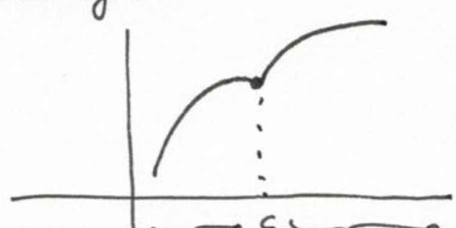
Remark: If $f'(c)$ does not exist, then $f''(c)$ also doesn't exist, and we can't use the 2nd derivative test. Then, we have to check if f' changes sign at c.

Remark: An inflection point can sometimes also be a crit. pt, but not always.



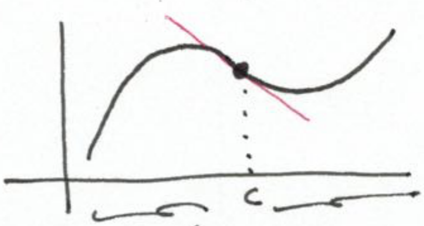
conc down conc up

c crit pt, inflection pt.



conc down conc down

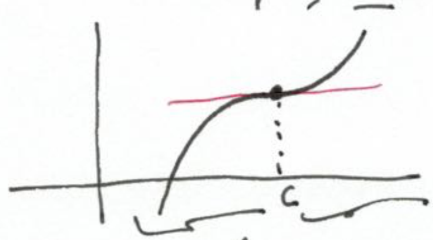
c crit pt, not an inflection pt



conc down conc up

c not crit pt: $f'(c) \neq 0$

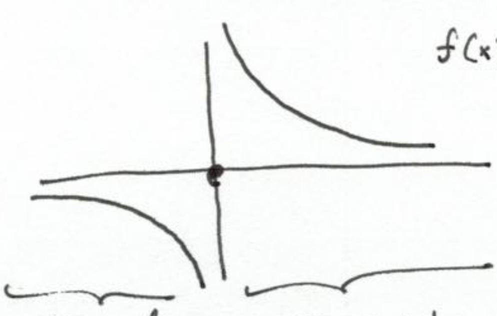
c is an inflection pt.



conc down conc up.

c crit pt : $f'(c) = 0$

c is an inflection pt.



conc down conc up

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

0 not an inflection pt
(not continuous)

crit pt (no derivative).

Ex $f(x) = x^4 - 4x^3, \quad -\infty < x < \infty$.

Draw the graph of f (min, max, concavity, inflection pts).

Critical points: f differentiable everywhere $\Rightarrow$ all crit pts $f'(x)=0$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \Rightarrow \underline{x=0}, \ \underline{x=3}.$$

Local min/max or neither? Because f'' exists everywhere $\Rightarrow$ 2nd deriv. test

$\underline{x=0}$: $f''(x) = 12x^2 - 24x = 12x(x-2)$

$f''(0) = 0$ no information if local min, max, or neither

$\Rightarrow$ use 1st derivative test, does f' change sign?

$f'(x) = 4x^2(x-3)$

near $x=0$: $x>0$ but small: $\underbrace{4x^2}_{>0}\underbrace{(x-3)}_{<0} < 0$

$x<0$ but small: $\underbrace{4x^2}_{>0}\underbrace{(x-3)}_{<0} < 0$

f' does not change sign $\Rightarrow 0$ is neither local min nor local max.

$\underline{x=3}$: $f''(3) = 12\cdot 3\underbrace{(3-2)}_{1} = 36 > 0$ concave up.

$\Rightarrow x=3$ is a local min.

Increasing/decreasing.

Interval	$f'(x)$	
$(-\infty, 0)$	<0	decreasing.
$(0, 3)$	<0	decreasing.
$(3, \infty)$	>0	increasing.

Inflection pts: $f''(x) = 12x(x-2) = 0 \Rightarrow \overbrace{\underline{x=0}, \ \underline{x=2}}^{\text{candidates for inflection pts.}}$

check if f'' changes sign.

$\underline{x=0}$: $x>0$ but small: $f''(x) = \underbrace{12x}_{>0}\underbrace{(x-2)}_{<0} < 0$

$x<0$ but small: $f''(x) = \underbrace{12x}_{<0}\underbrace{(x-2)}_{<0} > 0$

$\Rightarrow x=0$ is an inflection pt.

$\underline{x=2}$: if $x > 2$ but near 2 : $f''(x) = \underbrace{12x}_{>0}\underbrace{(x-2)}_{>0} > 0$

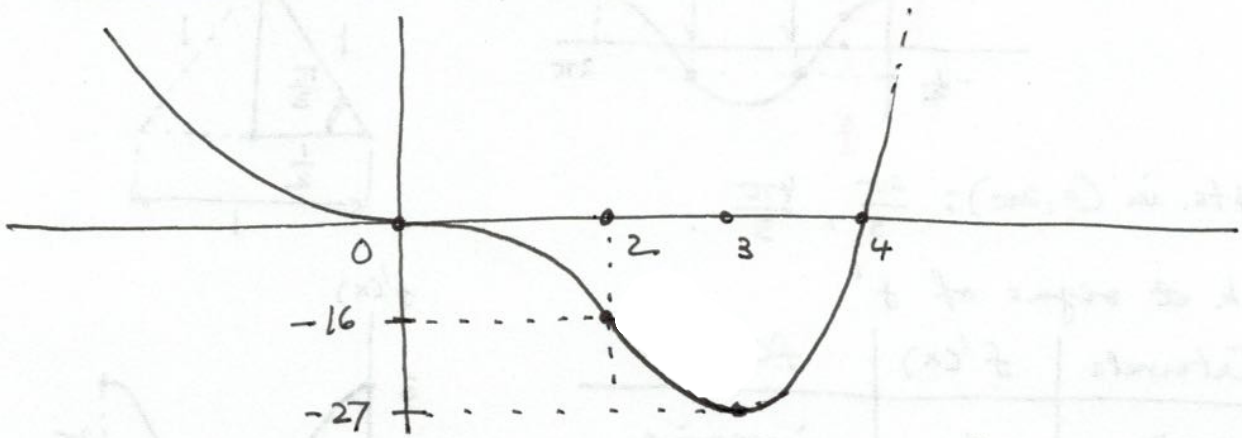
if $x < 2$ but near 2 : $f''(x) = \underbrace{12x}_{>0}\underbrace{(x-2)}_{<0} < 0$

$\Rightarrow$ $x = 2$ is an <u>inflection pt</u>.

<u>Concavity</u> (sign of f'').

Intervals	$f''(x) = 12x(x-2)$	concavity
$(-\infty, 0)$	> 0	up.
$(0, 2)$	< 0	down.
$(2, \infty)$	> 0	up.

<u>Draw the graph.</u> $f(x) = x^4 - 4x^3 = x^3(x-4)$.

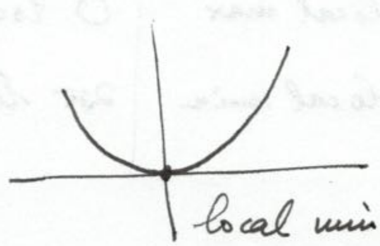


Crit pt: $f(3) = -27$, $f(2) = -16$.

concave up. | conc. down | concave up.

<u>Ex</u> $f(x) = x^4$. $f'(0) = 0$

$f''(0) = 0$



local min at 0