$\underline{\underline{Ex}}$: $$\int_0^1 \underbrace{x^3 e^{x^4}} dx = \frac{1}{4}e^{x^4}\Big|_0^1 = \frac{1}{4}e^1 - \frac{1}{4}e^0 = \frac{1}{4}(e-1)$$

$$f(x) = x^3 e^{x^4}$$

$$F(x) = \frac{1}{4}e^{x^4}$$ (since we can pick an arbitrary anti-derivative, we can let $C=0$)

$\underline{\underline{Ex}}$: $$\int_0^{\pi/4} \underbrace{\frac{1}{\cos^2 x}} dx = \tan x \Big|_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1.$$

$$f(x) = \frac{1}{\cos^2 x}.$$

$$F(x) = \tan x = \frac{\sin x}{\cos x}. \quad \underline{\text{Check}}: F'(x) = \frac{\overbrace{\cos x \cos x - \sin x(-\sin x)}^{\cos^2 x + \sin^2 x = 1}}{\cos^2 x}.$$

$\underline{\underline{Ex}}$: $$\int_a^b \underbrace{x^r} dx = \frac{x^{r+1}}{r+1}\Big|_a^b = \frac{b^{r+1} - a^{r+1}}{r+1}.$$

$$f(x) = x^r$$

$$F(x) = \frac{x^{r+1}}{r+1}$$

$\underline{\underline{Ex}}$: $$\int_0^{\pi/4} \underbrace{\frac{\sin x}{\cos x}} dx = -\ln(\cos x)\Big|_0^{\pi/4} = -\Big(\ln(\overbrace{\cos\tfrac{\pi}{4}}^{\frac{\sqrt{2}}{2}}) - \underbrace{\ln(\underbrace{\cos 0}_{1})}_{0}\Big)$$

$$= -\ln\frac{\sqrt{2}}{2} > 0.$$

$$f(x) = \frac{\sin x}{\cos x}.$$

$$F(x) = -\ln(\cos x) \quad \underline{\text{Check}}: F'(x) = \frac{-1}{\cos x}(-\sin x)$$

chain rule

Indefinite integrals.

Convenient notation for the most general antiderivative.

$$\int f(x)\,dx = F(x) + C$$

$\underline{\underline{Ex}}$ $$\int (x^3 + x)\,dx = \frac{x^4}{4} + \frac{x^2}{2} + C$$

$\underline{\underline{Note}}$: A definite integral $\int_a^b f(x)dx$ is a <u>number</u> (an area)

An indefinite integral $\int f(x)dx$ is a <u>function</u> (an antiderivative)

Properties: $\int c f(x)\,dx = c \int f(x)\,dx$

$$\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx .$$

Ex: $\int \sin x\,dx = -\cos x + C$

$$\int \frac{\cos x}{\sin^2 x}\,dx = -\frac{1}{\sin x} + C \quad \left(\text{check: } \left(\frac{-1}{\sin x}\right)' \underset{\text{quotient rule}}{=} \frac{0-(-1)\cdot\cos x}{\sin^2 x} = \frac{\cos x}{\sin^2 x}\right)$$

$$\int 2^x\,dx = \frac{2^x}{\ln 2} + C$$

$$\int \frac{1}{x}\,\frac{1}{(\ln x)^3}\,dx = -\frac{1}{2(\ln x)^2} + C \quad \left(\text{check: } \left(-\frac{1}{2}(\ln x)^{-2}\right)' = -\cancel{\tfrac{1}{2}}\,\cancel{(-2)}\,(\ln x)^{-3}\cdot\frac{1}{x}\right).$$

Interpretation of FToC, Part II:

Thm (Net change thm).

$$\underbrace{\int_a^b f'(x)\,dx}_{\text{rate of change, integrated from a to b.}} = f(x)\Big|_a^b = \underbrace{f(b) - f(a)}_{\text{total change of f going from a to b}} .$$

The substitution rule.

Helps to solve an integral whereby "the chain rule is reversed".

$$\int g'(f(x))\cdot f'(x)\,dx = g(f(x)) + C$$

because by the chain rule, $\big(g(f(x))\big)' = g'(f(x))\cdot f'(x)$.

$$\int_a^b g'(f(x))\cdot f'(x)\,dx = g(f(x))\Big|_a^b = g(f(b)) - g(f(a))$$

$\underline{\underline{Ex}}$: $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx = \int g'(f(x))\cdot f'(x)\,dx = g(f(x)) + C = -\frac{1}{\sin x} + C$

$f(x) = \sin x$, $f'(x) = \cos x$.

$g'(x) = \dfrac{1}{x^2} \Rightarrow g(x) = -\dfrac{1}{x}$

$\underline{\underline{Ex}}$: $\displaystyle\int_2^{10} \frac{1}{x}\frac{1}{(\ln x)^3}\,dx = \int_2^{10} g'(f(x))\, f'(x)\,dx = g(f(x))\Big|_2^{10} = -\frac{1}{2(\ln x)^2}\Big|_2^{10}$

$f(x) = \ln x$, $f'(x) = \dfrac{1}{x}$

$g'(x) = \dfrac{1}{x^3} \Rightarrow g(x) = -\dfrac{1}{2x^2}$

$$= -\left(\frac{1}{2(\ln 10)^2} - \frac{1}{2(\ln 2)^2}\right).$$

$$\int x^{-3}\,dx \underset{\uparrow}{=} \int x^r\,dx = \frac{x^{r+1}}{r+1} + C. = \frac{x^{-2}}{-2} + C$$

$r = -3$

$$= \frac{1}{-2x^2} + C$$

$\underline{\underline{Ex}}$: $\displaystyle\int_1^2 x^3 e^{x^4}\,dx = \int_1^2 g'(f(x))\cdot f'(x)\,dx = g(f(x))\Big|_1^2 = \frac{1}{4}e^{\cancel{4}\cdot\frac{x^4}{\cancel{4}}}\Big|_1^2$

$f(x) = \dfrac{x^4}{4}$, $f'(x) = x^3$

$g'(x) = e^{4x} \Rightarrow g(x) = \dfrac{1}{4}e^{4x}$

$$= \frac{1}{4}\left(e^{16} - e^1\right).$$

$g'(f(x)) = e^{4\cdot\frac{x^4}{4}}$

<u>Substitution rule:</u>

$$\int g'(\underbrace{f(x)}_{=:u})\cdot \underbrace{f'(x)\,dx}_{du}$$

$u = f(x)$.

$du = f'(x)\,dx$

(differentials).

$$= \int g'(u)\,du = g(u) + C$$

$$= g(f(x)) + C$$