Substitution rule:

$$\int g'(\underbrace{f(x)}_{u})\,\underbrace{f'(x)\,dx}_{du} = \int g'(u)\,du = g(u) + C .$$

Ex $\int \sin^7 x \cos x \, dx = \int u^7 \, du = \frac{u^8}{8} + C = \frac{\sin^8 x}{8} + C.$

$u = \sin x$
$du = \cos x \, dx$

Ex $\int x^5 \cos(x^6+1)\, dx = \int \frac{1}{6} \cos u \, du = \frac{1}{6} \sin u + C$

$= \frac{1}{6} \sin(x^6+1) + C$

$u = x^6+1$
$du = 6x^5 \, dx \Rightarrow x^5 dx = \frac{1}{6} du$

Ex $\int \sqrt{2x+1} \, dx = \int \sqrt{u} \, \frac{1}{2} du = \frac{1}{2} \frac{u^{3/2}}{3/2} + C = \frac{1}{3} u^{3/2} + C$

$= \frac{1}{3}(2x+1)^{3/2} + C$

$u = 2x+1$
$du = 2dx \Rightarrow dx = \frac{1}{2} du$

Ex $\int e^{5x+2} dx = \int e^u \frac{1}{5} du = \frac{1}{5} e^u + C = \frac{1}{5} e^{5x+2} + C.$

$u = 5x+2$
$du = 5\,dx \Rightarrow dx = \frac{1}{5} du$

Ex $\int \sqrt{1+x^2}\; x^5 dx = \int \sqrt{1+x^2}\,(x^2)^2\, x\, dx = \int \sqrt{u}\,(u-1)^2 \frac{1}{2} du$

$u = 1+x^2 \Rightarrow x^2 = u-1$
$du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2} du$

$$= \int \underbrace{\sqrt{u}}_{u^{1/2}} (u^2 - 2u + 1)\frac{1}{2} du = \int \left(u^{5/2} - 2u^{3/2} + u^{1/2}\right)\frac{1}{2} du$$

$$= \frac{1}{2}\left(\frac{2}{7} u^{7/2} - 2 \cdot \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2}\right) + C.$$

$$= \frac{1}{7} u^{7/2} - \frac{2}{5} u^{5/2} + \frac{1}{3} u^{3/2} + C$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C.$$

Substitution rule for definite integrals.

Thm

$$\int_a^b g'(\underbrace{f(x)}_{u})\,\underbrace{f'(x)\,dx}_{du} = g(f(x))\Big|_a^b = g(f(b)) - g(f(a)).$$

$$\int_{f(a)=u(a)}^{f(b)=u(b)} g'(u)\,du = g(u)\Big|_{f(a)=u(a)}^{f(b)=u(b)} = g(f(b)) - g(f(a))$$

Ex

$$\int_0^4 \sqrt{2x+1}\,dx = \int_1^9 \sqrt{u}\,\frac{1}{2}\,du = \cancel{\frac{1}{2}}\,\frac{\cancel{2}}{3}\,u^{3/2}\Big|_1^9 = \frac{1}{3}\left(\overbrace{9^{3/2}}^{27} - 1^{3/2}\right) = \frac{1}{3}\,26.$$

$u = 2x+1 \Rightarrow u(0) = 1,\ u(4) = 9$

$du = 2\,dx \Rightarrow dx = \frac{1}{2}\,du$

$$9^{3/2} = (\underbrace{9^{1/2}}_{3})^3 = 3^3 = 27$$

Ex

$$\int_1^2 \frac{1}{(3-5x)^2}\,dx = \int_{-2}^{-7} \frac{1}{u^2}\,\frac{-1}{5}\,du.$$

$u = 3-5x,\ u(1) = -2,\ u(2) = -7$

$du = -5\,dx \Rightarrow dx = \frac{-1}{5}\,du$

$-7 \quad -2 \quad 0$

Thm

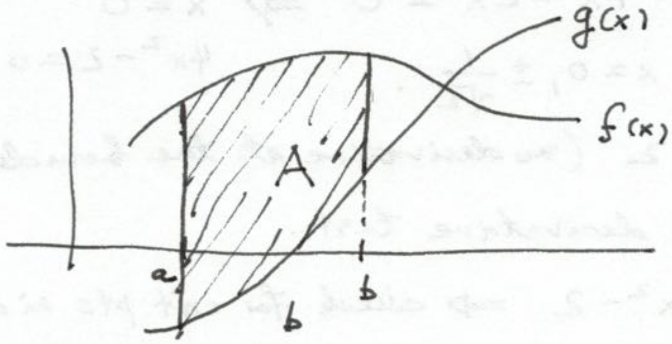
$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

$$= -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\,du = \frac{1}{5}\int_{-7}^{-2} \frac{1}{u^2}\,du$$

$$= \frac{1}{5}\left(\frac{-1}{u}\right)\Big|_{-7}^{-2} = \frac{1}{5}\left(\frac{-1}{-2} - \left(\frac{-1}{-7}\right)\right).$$

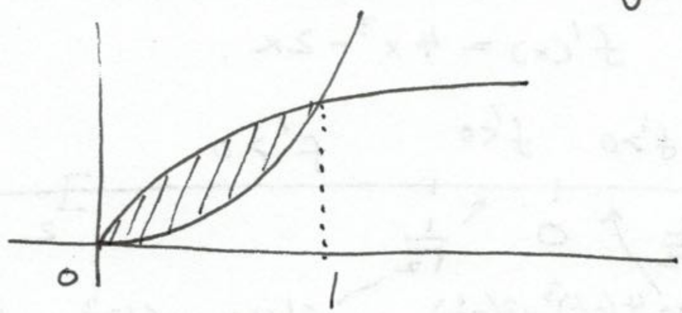
$$= \frac{1}{5}\left(\frac{1}{2} - \frac{1}{7}\right) = \cancel{\frac{1}{5}}\,\frac{\cancel{7-2}}{14} = \frac{1}{14}.$$

Areas between curves.



$$A = \int_a^b \big(f(x) - g(x)\big)\,dx$$

Ex Find the area enclosed by $y = x^2$ and $y = 2x - x^2$.



intersection point:

$x^2 = 2x - x^2$.

$\Rightarrow \cancel{2}x^2 = \cancel{2}x \quad \Rightarrow \underline{x = 0}$.

$\Rightarrow \underline{\underline{x = 1}}$

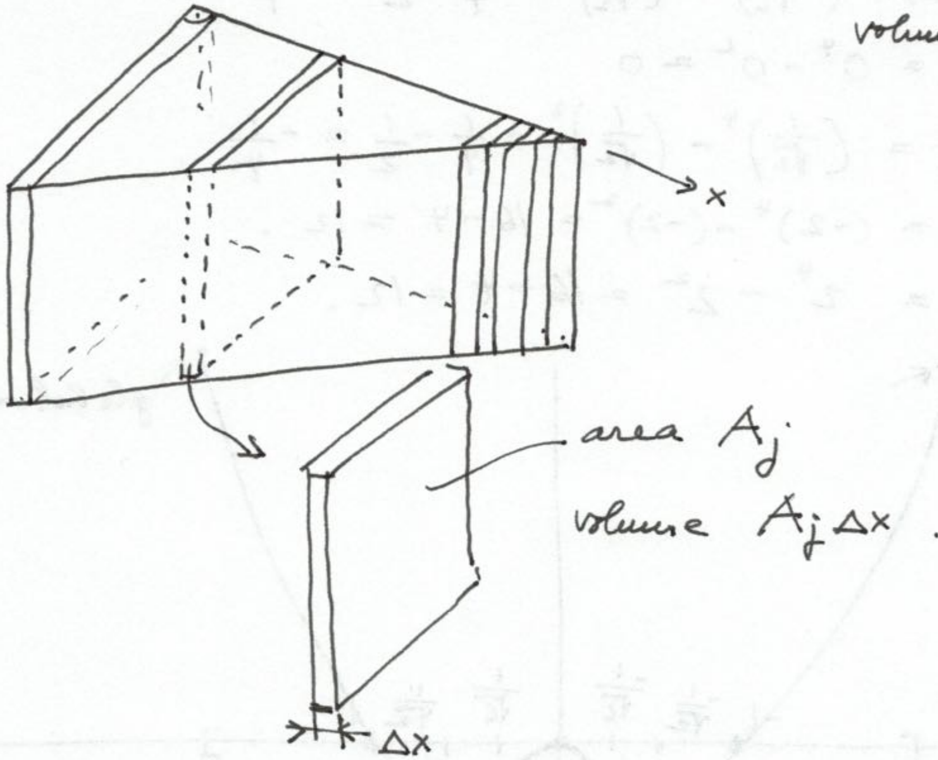
$$A = \int_0^1 \left(2x - x^2 - x^2\right) dx$$

$$= \int_0^1 \left(2x - 2x^2\right) dx = \left(2 \cdot \frac{x^2}{2} - 2 \cdot \frac{x^3}{3}\right)\Big|_0^1$$

$$= \left(1^2 - \frac{2}{3} \cdot 1^3 - 0\right) = \underline{\underline{\frac{1}{3}}}.$$

Volumes.

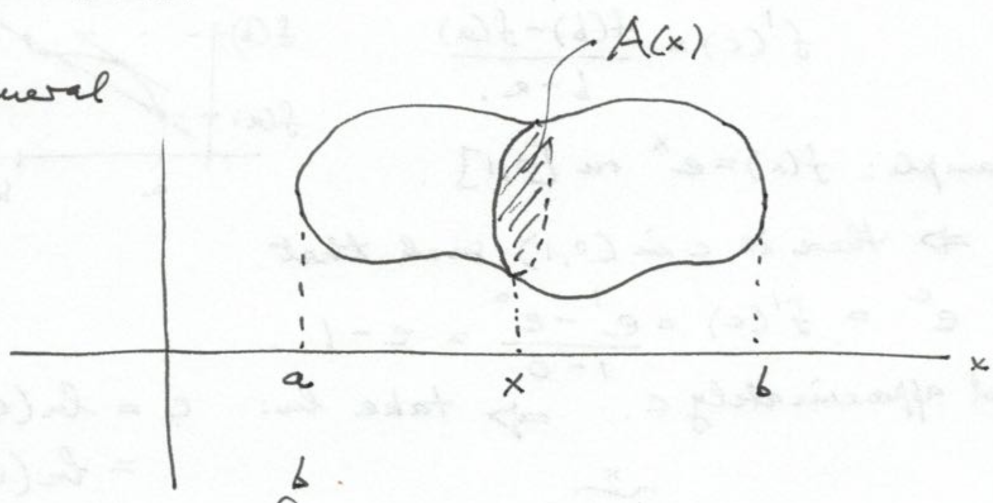
volume of cheese.



$$\text{Volume} = \underbrace{A_1 \Delta x + A_2 \Delta x + \dots + A_n \Delta x}_{\text{Riemann sum}}.$$

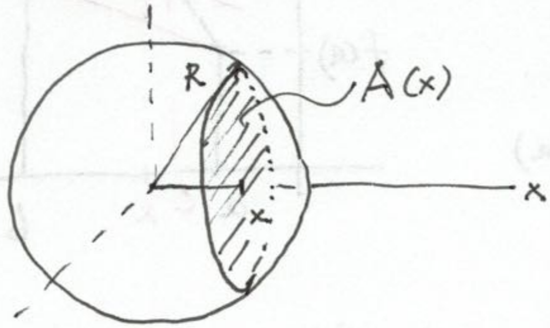
By taking Δx smaller and n larger (more slices), we get in the limit where $\Delta x \to 0$, $n \to \infty$, an integral via a Riemann sum.

In general

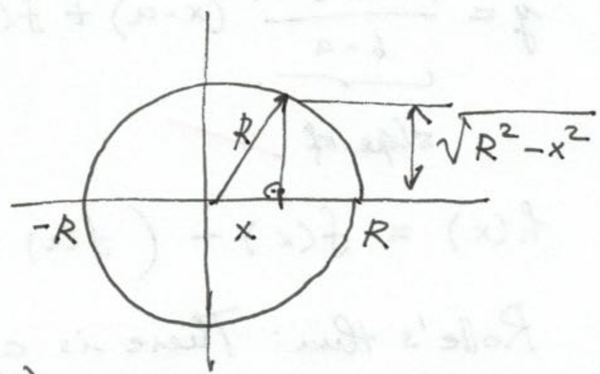


$$\text{Volume} = \int_a^b A(x)\,dx$$

Ex Find the volume of a ball of radius R.



Find area of $A(x)$.



$A(x)$ is a round disc with radius $\sqrt{R^2-x^2}$

$$\Rightarrow \quad A(x) = \pi\left(\sqrt{R^2-x^2}\right)^2 = \pi\left(R^2-x^2\right)$$

$$\text{Volume of ball} = \int_{-R}^{R} A(x)\,dx = \int_{-R}^{R} \pi\left(R^2-x^2\right)dx$$

$$= \pi\left(R^2x - \frac{x^3}{3}\right)\Bigg|_{-R}^{R}$$

$$= \pi\left(R^2\cdot R - \frac{R^3}{3}\right) - \pi\left(R^2(-R) - \frac{(-R)^3}{3}\right).$$

$$= \pi R^3\,\frac{2}{3} + \pi\left(R^3 - \frac{1}{3}R^3\right).$$

$$= \pi R^3\,\frac{2}{3} + \pi R^2\,\frac{2}{3}.$$

$$= \frac{4\pi}{3}R^3.$$

Rate of change of volume of ball with change of radius:

$$V'(R) = \frac{4\pi}{3}\cdot 3R^2 = 4\pi R^2$$

surface area.