Cyclic vectors in Dirichlet-type spaces

Constanze Liaw (Baylor University)

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Broader Impacts of the problem of cyclicity

- Invariant subspace problem and cyclic vectors:
  Does every bounded operator $T$ on a Hilbert space $\mathcal{H}$ have a non-trivial closed invariant subspace (i.e. $T(W) \subset W$)?
  NO, IF one can find an operator $T$ such that every $0 \neq \varphi \in \mathcal{H}$ is cyclic (i.e. $\mathcal{H} = \text{clos span}\{T^n\varphi : n \in \mathbb{N}\}$).

- Structure (basic building blocks) of a function space determined by its cyclic vectors

- Brown–Shields conjecture

- For physicists, the cyclicity of an operator means that the spectrum has multiplicity one
One complex variable
Consider the *Dirichlet-type spaces* $\mathcal{D}_\alpha$, i.e. bounded analytic functions on the unit disk $\mathbb{D} \subset \mathbb{C}$ with norm 
$$
\| f \|_{\mathcal{D}_\alpha}^2 = \sum_{k=0}^{\infty} (k + 1)^\alpha |a_k|^2 < \infty, \text{ where } f(z) = \sum_{k=0}^{\infty} a_k z^k
$$

- Bergman $A^2 = \mathcal{D}_{-1}$; Hardy $H^2 = \mathcal{D}_0$; and Dirichlet $\mathcal{D} = \mathcal{D}_1$
- A vector $f$ is cyclic (under the forward shift) for $\mathcal{D}_\alpha$ if 
$$
\mathcal{D}_\alpha = \text{span}\{z^k f(z) : k \in \mathbb{N} \cup \{0\}\}
$$
- The constant function $1$ is cyclic for $\mathcal{D}_\alpha$
- $f \in \mathcal{D}_\alpha$ cyclic, implies $f(z) \neq 0$ for $z \in \mathbb{D}$

“The fewer zeros the easier is cyclicity.”
Optimality

- Note $f$ is cyclic in $D_\alpha$ iff
  \[ N_n(f, \alpha) := \inf_{p_n} \| p_n f - 1 \|^2_{D_\alpha} \to 0 \quad \text{as } n \to \infty \]

- If $f(z) = 1 - z$, then $p_n = (\text{order } n \text{ Taylor poly. of } 1/f)$ yields
  \[ \| p_n f - 1 \|^2_{D_\alpha} = n + 2 \]

Two types of results:
- Optimal sequence of polynomials $p_n$
- The optimal rate of decay of these norms $N_n(f, \alpha)$ as $n \to \infty$
Example of explicit optimal approximants

For \( f(z) = 1 - z \), optimal for

\[ H^2 : \quad C_n(z) = \sum_{k=0}^{n} \left( 1 - \frac{k}{n+1} \right) z^k, \]

\[ D : \quad R_n(z) = \sum_{k=0}^{n} \left( 1 - \frac{H_{k+1}}{H_{n+2}} \right) z^k, \quad H_n = \sum_{k=2}^{n} \frac{1}{k}, \]

\[ A^2 : \quad S_n(z) = \sum_{k=0}^{n} \left( 1 - \frac{k(k+3)}{(n+1)(n+4)} \right) z^k. \]
Rate of decay

Let $H_n = \sum_{k=2}^{n} \frac{1}{k}$ and note that $H_n \approx \log n$ for large $n$.

**Definition**

For $\alpha < 1$, we set $\phi_\alpha(n) = n^{\alpha-1}$, $n \in \mathbb{N}$.
For $\alpha = 1$, we use $\phi_1(n) = 1/H_n$, $n \in \mathbb{N}$.

**Theorem (Béníeteau–Condori–L.–Seco–Sola, J. d’A. accepted)**

Suppose $f \in D_\alpha$, $\alpha \leq 1$, can be extended analytically to some strictly bigger disk. Suppose also that $f$ does not vanish in $\mathbb{D}$.
Then there exists a constant $C_0$ so that the optimal norm satisfies

$$N_n(f, \alpha) \leq C_0 \phi_\alpha(n + 1).$$

Moreover, for polynomial $f$ with zero on $\mathbb{T}$, and $\alpha = 1, 0, -1$, there is a constant $C_1$ so that

$$C_1 \phi_\alpha(n + 1) \leq N_n(f, \alpha).$$

Polynomials that have no zeros in $\mathbb{D}$ are cyclic in $D_\alpha$ for $\alpha \leq 1$. 
Partial result on the Brown–Shields conjecture
Outer

- Vectors in $H^2$ are cyclic iff they are outer
- For $\alpha \geq 0$: If $f$ cyclic in $D_\alpha$, then $f$ outer

Logarithmic capacity

- Non-tangentially $f^*(\zeta) = \lim_{z \to \zeta \in \mathbb{T}} f(z)$
- For $f \in D$, $f^*$ exists outside a set of logarithmic capacity zero
- Zero set $Z(f) = \{ \zeta \in \mathbb{T} : f^*(\zeta) = 0 \}$
- Brown–Shields: If $f \in D$ is cyclic, then $Z(f)$ has capacity zero


A vector $f \in D$ is cyclic iff it is outer and has $Z(f)$ capacity zero.

Brown–Cohn: For any closed set of logarithmic capacity zero $E \subset \mathbb{T}$, there exists a cyclic function $f$ in $D$ with $Z(f) = E$. 
Two weak versions of the Brown–Shields conjecture:


A vector $f \in D$ is cyclic, if it is outer and $Z(f)$ is countable.

**Theorem (El-Fallah–Kellay–Ransford 2006)**

The condition ‘countable’ can be replaced by one which is closer to ‘capacity zero’, but VERY complicated.
Theorem (Bénéteau–Condori–L.–Seco–Sola, J. d’A. accepted)

Suppose \( f \in \mathcal{D} \) and \( \log f \in \mathcal{D} \). Then \( f \) is cyclic in \( \mathcal{D} \).

Theorem (Bénéteau–Condori–L.–Seco–Sola, J. d’A. accepted)

Let \( f \in H^\infty \) and \( q = \log f \in \mathcal{D}_\alpha, \alpha \leq 1 \). Suppose there exist polynomials \( q_n \) of degree \( \leq n \) that approach \( q \) in \( \mathcal{D}_\alpha \) norm with

\[
\sup_{z \in \mathbb{D}} \text{Re}(q(z) - q_n(z)) + \log \|q_n - q\| \leq C
\]

for some constant \( C > 0 \). Then \( f \) is cyclic in \( \mathcal{D}_\alpha \).

Brown–Cohn’s examples satisfy above assumptions.
Two complex variables
Dirichlet-type space on the bidisk

- **Bidisk** $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$

- Holomorphic $f : \mathbb{D}^2 \to \mathbb{C}$ belongs to the *Dirichlet-type space* $\mathcal{D}_\alpha$ if its power series $f(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} z_1^k z_2^l$ satisfies

  $$\|f\|_{2,\alpha}^2 = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (k+1)^\alpha (l+1)^\alpha |a_{k,l}|^2 < \infty$$

- Function $f \in \mathcal{D}_\alpha$ is *cyclic*, if

  $$\mathcal{D}_\alpha := \text{span}\{z_1^k z_2^l f : k = 0, 1, \ldots; l = 0, 1, \ldots\}$$

- Let $\mathcal{P}_n, n \in \mathbb{N}$, be the polynomials of the form

  $$p_n = \sum_{k=0}^{n} \sum_{l=0}^{n} c_{k,l} z_1^k z_2^l$$

- $f$ is cyclic iff $\mathcal{N}_n(f, \alpha) := \inf_{p_n \in \mathcal{P}_n} \|p_n f - 1\|_{2,\alpha} \xrightarrow{n \to \infty} 0$
Reductions to functions of one variable
Reduction to functions of one variable

Consider

\[
J_{\alpha,M,N} := \left\{ f \in D_{\alpha} : f = \sum_{k=0}^{\infty} a_k z_1^{Mk} z_2^{Nk} \right\},
\]

e.g. \( f(z_1, z_2) = 1 - z_1 z_2 \in J_{\alpha,1,1} \)

Consider the mappings

\[
L_{M,N} : D_{2\alpha} \to D_{\alpha} \quad \text{via} \quad L_{M,N}(F)(z_1, z_2) = F(z_1^M \cdot z_2^N),
\]

\[
R_{M,N} : J_{\alpha,M,N} \to D_{2\alpha} \quad \text{via} \quad R_{M,N}(f)(z) = f(z^{1/M}, 1)
\]

If \( f \in J_{\alpha,M,N} \), there exist constants such that

\[
c_2 \| R(f) \|_{D_{2\alpha}} \leq \| f \|_\alpha \leq c_1 \| R(f) \|_{D_{2\alpha}}
\]

Note the change from \( D_{\alpha} \) for bidisk to \( D_{2\alpha} \) for disk!
Let \( f \in J_{\alpha,M,N} \) have the property that \( R(f) = f(z^{1/M}, 1) \) is a function that admits an analytic continuation to the closed unit disk, whose zeros lie in \( \mathbb{C} \setminus \mathbb{D} \).

Then \( f \) is cyclic in \( \mathcal{D}_\alpha \), and there exists a constant \( C = C(\alpha, f, M, N) \) such that

\[
\mathcal{M}_n(f, \alpha) \leq C \varphi_{2\alpha}(n + 1).
\]

This result is sharp in the sense that, if \( R(f) \) has at least one zero on \( \mathbb{T} \), then there exists \( c = c(\alpha, f, M, N) \) such that for large \( n \):

\[
c \varphi_{2\alpha}(n + 1) \leq \mathcal{M}_n(f, \alpha).
\]

Here \( \varphi_{2\alpha}(n) = \left\{ \begin{array}{ll}
n^{2\alpha - 1} & \text{for } 2\alpha < 1 \\
1/ \sum_{k=2}^{n} \frac{1}{k} & \text{for } 2\alpha = 1
\end{array} \right\} \) increases if \( \alpha > 1/2 \).
Examples

- Functions like $f(z_1, z_2) = 1 - z_1$, $f(z_1, z_2) = (1 - z_1 z_2)^N$, $N \in \mathbb{N}$, and $f(z_1, z_2) = z_1^2 z_2^2 - 2(\cos \theta) z_1 z_2 + 1$, $\theta \in \mathbb{R}$, satisfy the assumptions of the theorem.

- Polynomial $g(z_1, z_2) = 1 - z_1 z_2$ is not cyclic in $\mathcal{D}_\alpha$ for $\alpha > 1/2$, although it is only zero for $z_1 = z_2 = 1$.

- Notice that $g$ is outer, but its zero set \{ $z_1 = z_2 = 1$ \} has non-zero logarithmic capacity.
Open problems

- The *Brown-Shields conjecture* for functions on the bidisk: Is the condition that \( f \in \mathcal{D} \) is outer and the zero set of \( f \) (on the boundary) has logarithmic capacity 0 sufficient for \( f \) to be cyclic?

- Sub-problem: Characterize the *cyclic polynomials* \( f \in \mathcal{D}_\alpha \) for each \( \alpha \leq 1 \).