Approximate Analysis to the KdV-Burgers Equation

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Outline

1 Introduction
   - Generalized KdV-Burgers Equation
   - KdV-Burgers Equation
   - Planar Polynomial Systems and Abel Equation
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   • Generalized KdV-Burgers Equation
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2 Qualitative Analysis
   • Generalized Abel Equation
   • Property of Our Operator
   • Two Theorems
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2 Qualitative Analysis
- Generalized Abel Equation
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3 Approximate Solution
- 2D KdV-Burgers Equation
- Resultant Abel Equation
- Approximate Solution to 2D KdV-Burgers Equation
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   - Generalized KdV-Burgers Equation
   - KdV-Burgers Equation
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   - Resultant Abel Equation
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5. Acknowledgement
Generalized KdV-Burgers Equation

- Generalized Korteweg-de Vries-Burgers equation [1, 2]

\[ u_t + \left( \delta u_{xx} + \frac{\beta}{p} u^p \right)_x + \alpha u_x - \mu u_{xx} = 0, \]  

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where \( u \) is a function of \( x \) and \( t \), \( \alpha, \beta \) and \( p > 0 \) are real constants, \( \mu \) and \( \delta \) are coefficients of dissipation and dispersion, respectively.
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Burgers Equation and KdV Equation

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- Solitary wave solutions of equation (4) are as follows [7, 8, 9]:

$$u(x, t) = \frac{3\beta^2}{25\alpha s} \text{sech}^2 \Psi - \frac{6\beta^2}{25\alpha s} \tanh \Psi \pm \frac{6\beta^2}{25\alpha s}, \quad (5)$$
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Figures of Wave Solutions

Legend
- u1-Burgers-KdV
- u2-Burgers-KdV
- u3-Burgers
- u4-KdV
Planar Polynomial Systems and Abel Equation

- Consider planar polynomial systems of the form

\[ \dot{x} = -y + p(x, y), \quad \dot{y} = x + q(x, y) \]  

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For the Poincaré center problem, setting \( x = r \cos \theta, \ y = r \sin \theta \) gives
\[ \frac{dr}{d\theta} = \frac{r^k \xi(\theta)}{1 + r^{k-1} \eta(\theta)}, \] (7)
where \( \xi \) and \( \eta \) are polynomials in \( \cos \theta \) and \( \sin \theta \) of degree \( k + 1 \).
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- Making the coordinate transformation
  \[ \rho = \frac{r^{k-1}}{1 + r^{k-1} \eta(\theta)}, \]
  we get an Abel equation
  \[ \frac{d\rho}{d\theta} = a(\theta) \rho^2 + b(\theta) \rho^3, \]
  where \( a = (k - 1) \xi + \eta' \) and \( b = (1 - k) \xi \eta \).
Traveling Wave Solution

- Assume that equation (1) has the traveling wave solution of the form

\[ u(x, t) = u(\xi), \quad \xi = x - ct, \]
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where \( c \neq 0 \) is the wave velocity. Then equation (1) becomes

\[ \delta u''' - \mu u'' + (\alpha - c)u' + \beta u^{p-1}u' = 0, \tag{8} \]
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\[ u'' - gu' - eu - fu^p - d = 0, \]
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- Assume that \( y = u \) and \( u' = z \), then equation (9) is equivalent to

\[ \begin{cases} 
  y' = z, \\
  z' = ey + gz + fy^p + d.
\end{cases} \quad (10) \]
Global Structure of $p = 2$
It follows from system (10) that

\[
\frac{dz}{dy} = \frac{ey + gz + fy^p + d}{z}.
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(11)
Transformed to Abel Equation

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- Question: Under what condition one can determine the number of closed solutions of the Abel equation (12).
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Open Problem: There have been two longstanding problems, called the Poincaré center-focus problem and the local Hilbert 16th problem. Both are closely related to the Bautin quantities and the Bautin ideal of the Abel equation.
Consider the generalized Abel equation

\[ r' = a(t)r^2 + b(t)r^n, \quad r(t_0) = c, \quad t \in [t_0, t_1], \quad n \geq 3. \quad (13) \]
Integral Form

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Dividing both sides of equation (13) by \( r^2 \) gives

\[ \frac{r'}{r^2} = a(t) + b(t)r^{n-2}. \quad (14) \]
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Integrating equation (14) from \( t_0 \) to \( t \) yields
\[ r(t) = \frac{c}{1 - cA(t) - c \int_{t_0}^{t} b(\tau)r^{n-2}d\tau}, \quad (15) \]
where \( A(t) = \int_{t_0}^{t} a(\tau)d\tau. \)
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Integrating equation (14) from \(t_0\) to \(t\) yields
\[ r(t) = c \frac{1 - cA(t) - c \int_{t_0}^{t} b(\tau)r^{n-2}d\tau}{1 - cA(t) - c \int_{t_0}^{t} b(\tau)r^{n-2}d\tau}, \] (15)

where \(A(t) = \int_{t_0}^{t} a(\tau)d\tau.\)

Rewrite equation (15) as
\[ r(t) = c \left(1 + A(t) + r(t) \int_{t_0}^{t} b(\tau)r^{n-2}d\tau \right). \] (16)
Let $C[0, 1]$ denote the Banach space of all continuous functions on the interval $[0, 1]$ with the norm $\|f\| = \max_{0 \leq t \leq 1} |f(t)|$. We define the operator [10]:

$$T_c : C[0, 1] \to C[0, 1]$$
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A Nonlinear Operator

Let $\mathcal{C}[0, 1]$ denote the Banach space of all continuous functions on the interval $[0, 1]$ with the norm $\|f\| = \max_{0 \leq t \leq 1} |f(t)|$. We define the operator [10]:

$$T_c : \mathcal{C}[0, 1] \to \mathcal{C}[0, 1],$$

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for given $a, b \in \mathcal{C}[0, 1]$ and $c \in \mathbb{R}$. Obviously, $T_c$ is well defined on an arbitrary bounded set of $\mathcal{C}[0, 1]$ if $c$ is suitably small. Let us first observe some useful properties of $T_c$. 

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KdV-Burgers Equation  Z. Feng  Department of Mathematics, University of Texas-Pan American, Edinburg, USA
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Lemma (1)

For \( f \in C[0, 1] \) and \( c \in \mathbb{R} \) with \( \| f \| \leq M \) and \( |c| < c_0 \) defined as \( c_0 = (\| a \| + \| b \| M^{n-2})^{-1} \), \( T_c(f) \) is well defined and differentiable, and satisfies

\[
\frac{d}{dt} T_c(f)(t) = a(t)[T_c(f)(t)]^2 + b(t)[T_c(f)(t)]^2 f(t)^{n-2}.
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Furthermore, we have an identity

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T_c(f)(t) - T_c(g)(t) = T_c(f)(t)T_c(g)(t) \int_0^t b(\tau)(f(\tau)^{n-2} - g(\tau)^{n-2})d\tau,
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\( 0 \leq t \leq 1 \).
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for arbitrary \( f, g \in C[0, 1] \) and \( c \in \mathbb{R} \) with \( \|f\|, \|g\| \leq M \) and \( |c| < c_0 \).
Outline of the Proof

Step 1: well-defined

\[ 1 - cA(t) - c \int_{0}^{t} b(\tau)f(\tau)^{n-2}d\tau = 0 \Rightarrow \]

\[ |c| \geq \frac{1}{|A(t)| + \int_{0}^{t} |b(\tau)f(\tau)^{n-2}|d\tau} \geq \frac{1}{\|a\| + \|b\|M^{n-2}}. \]
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Step 2: A direct calculation gives

\[
\frac{d}{dt} T_c(f)(t) = \frac{-c[-ca(t) - cb(t)f(t)^{n-2}]}{(1 - cA(t) - c \int_0^t b(\tau)f(\tau)^{n-2}d\tau)^2} = \frac{c^2 a(t)}{(1 - cA(t) - c \int_0^t b(\tau)f(\tau)^{n-2}d\tau)^2} + \frac{c^2 b(t)f(t)^{n-2}}{(1 - cA(t) - c \int_0^t b(\tau)f(\tau)^{n-2}d\tau)^2}
\]

\[ T_c(f)(t) - T_c(g)(t) = \frac{c}{H(f)} \cdot \frac{c}{H(g)} \cdot \int_0^t b(\tau)(f(\tau)^{n-2} - g(\tau)^{n-2})d\tau \]
Lemma (2)

Let \( c_1 = (\|a\| + \|b\| + 1)^{-1} \). Then we have

\[
\| T_c f \| \leq 1 \quad \text{if} \quad \|f\| \leq 1 \quad \text{and} \quad |c| \leq c_1.
\]
Lemma 2

Lemma (2)

Let $c_1 = (\|a\| + \|b\| + 1)^{-1}$. Then we have

$$\|T_c f\| \leq 1 \quad \text{if} \quad \|f\| \leq 1 \quad \text{and} \quad |c| \leq c_1.$$

Outline of the Proof.

If $\|f\| \leq 1$ and $|c| \leq c_1$, then we have

$$\|T_c f\| \leq \frac{|c|}{1 - |c| (\|a\| + \|b\|\|f\|^{n-2})} \leq \frac{|c|}{1 - |c| (\|a\| + \|b\|)} \leq 1.$$

The conclusion follows.
Lemma 3

Lemma (3)

Let $c_2 = \left( \sqrt{(n - 2)} \| b \| + \| a \| + \| b \| + 1 \right)^{-1}$. If $|c| \leq c_2$, then $T_c$ is a contraction mapping on the close unit ball $B_1 = \{ f \in C[0, 1] \mid \| f \| \leq 1 \}$ of $C[0, 1]$.

Outline of the Proof.

It follows from Lemmas 1 and 2 that

$$\| T_c(f)(t) - T_c(g)(t) \| \leq \| T_c(f) \| \| T_c(g) \| \| b \| \| f^{n-2} - g^{n-2} \|$$

$$= C \| (f - g)(f^{n-3} + f^{n-4}g + \cdots + fg^{n-4} + g^{n-3}) \|$$

$$\leq (n - 2)c \| f - g \|,$$

where

$$c \overset{\text{def}}{=} \left( \frac{|c|}{1 - |c| (\| a \| + \| b \|)} \right)^2 \| b \|. $$
Theorem 1

**Theorem (1)**

For given \( a, b \in C[0, 1] \) and \( c \in \mathbb{R} \) with
\[
|c| \leq (\sqrt{n - 2} \|b\| + \|a\| + \|b\| + 1)^{-1},
\]
the solution \( r(t, c) \) of equation (1) with \( r(0, c) = c \) can be uniformly approximated by an iterated sequence \( \{T^n_c(f)(t)\} \):
\[
r(t, c) = \lim_{n \to \infty} T^n_c(f)(t), \quad 0 \leq t \leq 1, \tag{17}
\]
that is,
\[
r(t, c) = \frac{c}{1 - cA(t) - c^{n-1} \int_0^t \frac{b(t_1)dt_1}{1-cA(t_1)-c^{n-1} \int_0^{t_1} \frac{b(t_2)dt_2}{1-cA(t_2)-c^{n-1} \int_0^{t_2} \cdots}}}, \tag{18}
\]
for arbitrary \( f \in C[0, 1] \) with \( \|f\| \leq 1 \). Furthermore, the following error estimate holds
\[
r(t, c) - T^n_c(f)(t) = O(c^{2n}).
\]
Theorem 2: Case of \( n = 3 \)

- Denote

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M = \max_{t \in [0,1]} |a(t) \pm b(t)|. 
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Theorem (2)

Suppose $a, b \in C[0,1]$ and $c \in \mathbb{R}$ with

\[ |c| \leq \max\{ (\sqrt{\|b\|} + \|a\| + \|b\| + 1)^{-1}, (2M)^{-1} \}. \]

Then, in formula (18), the following part is bounded

\[
\frac{b(t_1)}{1 - cA(t_1) - c^2 \int_0^{t_1} \frac{b(t_2)dt_2}{1 - cA(t_2) - c^2 \int_0^{t_2} \cdots}} = \frac{1}{c} \cdot \frac{c}{1 - cA(t_1) - c^2 \int_0^{t_1} b(t_2) \cdot \frac{c}{1 - cA(t_2) - c^2 \int_0^{t_2} \cdots dt_2}}.
\]
Consider the 2D Korteweg-de Vries-Burgers equation:

\[(U_t + \alpha UU_x + \beta U_{xx} + sU_{xxx})_x + \gamma U_{yy} = 0,\]  

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Assume that equation (19) has an exact solution in the form

\[
U(x, y, t) = U(\xi), \quad \xi = hx + ly - wt.
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Assume that equation (19) has an exact solution in the form

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U(x, y, t) = U(\xi), \quad \xi = hx + ly - wt. \tag{20}
\]

Substitution of (20) into equation (19) and performing integration twice yields

\[
U''(\xi) + \lambda U'(\xi) + aU^2(\xi) + bU(\xi) + d = 0, \tag{21}
\]

where \( v = U(\xi) \in [v_0, v_1] \), \( \lambda = \frac{\beta}{sh} \), \( a = \frac{\alpha}{2sh^2} \), \( b = \frac{\gamma l^2 - wh}{sh^4} \) and \( d = -\frac{C}{sh^4} \).
Let $v = U(\xi)$ and $y = U'(\xi)$. Equation (21) becomes

$$\frac{dy}{dv}y + \lambda y + av^2 + bv + d = 0.$$  \hspace{1cm} (22)
Resultant Abel Equation

Let \( v = U(\xi) \) and \( y = U'(\xi) \). Equation (21) becomes

\[
\frac{dy}{dv} + \lambda y + av^2 + bv + d = 0. \tag{22}
\]

Using \( z = \frac{1}{y} \) yields

\[
\frac{dz}{dv} = \lambda z^2 + (av^2 + bv + d)z^3, \quad z(v_0) = \frac{1}{U'(\xi_0)} = c. \tag{23}
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- Let $\eta = \frac{v - v_0}{v_1 - v_0}$, then $\eta \in [0, 1]$ and $v = v_0 + (v_1 - v_0)\eta$. 
Let $v = U(\xi)$ and $y = U'(\xi)$. Equation (21) becomes

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$$r' = h(\eta)r^2 + k(\eta)r^3, \quad r(0) = c, \quad (24)$$
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r' = h(\eta)r^2 + k(\eta)r^3, \quad r(0) = c, \] (24)

where \( h(\eta), k(\eta) \in \mathcal{C}[0, 1] \), and

\[
h(\eta) = (v_1 - v_0)\lambda, \quad k(\eta) = (v_1 - v_0)(av^2 + bv + d). \]
Solution to Equation (24)

By virtue of Theorem 1, if $|c| \leq (\sqrt{\|k\|} + \|h\| + \|k\| + 1)^{-1}$, the solution to equation (24) is

$$r(\eta) = \lim_{n \to +\infty} T^n_c (w)(\eta),$$  \hspace{1cm} (25)
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where \(0 \leq \eta \leq 1\) for any \(w \in C[0, 1]\) with \(\|w\| \leq 1\), and

\[ T_c(w) = \frac{c}{1 - cH(\eta) - c \int_0^\eta k(x)w(x)^{n-2}dx} \]
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where

$$H(\eta) = \int_{0}^{\eta} h(x) \, dx = \int_{0}^{\eta} (v_{1} - v_{0}) \lambda \, dx = (v_{1} - v_{0}) \lambda \eta,$$

$$k(x) = (v_{1} - v_{0}) \left( a(v_{0} + (v_{1} - v_{0})x)^{2} + b(v_{0} + (v_{1} - v_{0})x) + d \right).$$
Recall that \( r = \frac{1}{y} \), \( y = U'(\xi) \), \( \eta = \frac{v-v_0}{v_1-v_0} \) and \( v = U(\xi) \). When conditions of Theorem 1 are fulfilled, we have

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\frac{1}{U'(\xi)} = \frac{c}{1 - cA(\xi) - c^2 \int_0^{\xi} \frac{b(t_1)dt_1}{1-cA(t_1)-c^2 \int_0^{t_1} \frac{b(t_2)dt_2}{1-cA(t_2)-c^2 \int_0^{t_2} \cdots}}.
\]
Approximate Solution to 2D-KdV-Burgers Equation

- Recall that $r = \frac{1}{y}$, $y = U' (\xi)$, $\eta = \frac{v - v_0}{v_1 - v_0}$ and $v = U (\xi)$. When conditions of Theorem 1 are fulfilled, we have

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- When $c$ is small, according to Theorem 2, the coefficient of $c^2$ is bounded. So we can drop the term containing $c^2$ and get

$$U' (\xi) \approx \frac{1 - c(v_1 - v_0) \lambda \eta}{c} \frac{c}{1 - c \lambda (U (\xi) - v_0)} = \frac{1 - c \lambda (U (\xi) - v_0)}{c}. \quad (27)$$
Recall that \( r = \frac{1}{y} \), \( y = U'(\xi) \), \( \eta = \frac{v-v_0}{v_1-v_0} \) and \( v = U(\xi) \). When conditions of Theorem 1 are fulfilled, we have

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\]

That is,

\[
U'(\xi) + \lambda U(\xi) = \frac{1}{c} + \lambda v_0.
\]

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Solving equation (27) gives

\[ U(x, y, t) = \frac{1}{c} + \frac{\lambda v_0}{\lambda} + ce^{-\lambda \xi}, \quad \xi = hx + ly - wt \]

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If we take \( v_0 = \frac{b}{2a} \) and choose \( c = \frac{-2a}{\lambda \sqrt{b^2 - 4ad}} \) sufficiently small, when \( \lambda \xi \to +\infty \), we obtain

\[ U(x, y, t) \sim \frac{b^2 - 4ad}{-2a} + \frac{b}{2a}. \]  

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- It is remarkable that the approximate solution (28) is in agreement with main results described in [7, 8] by the Hardy’s theory and the theory of Lie symmetry.
Approximate Solution to 2D KdV-Burgers Equation

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Note that equation (26) can be rewritten as

\[ \frac{1}{U'(\xi)} = \frac{c}{1 - cA(\xi) - c^2\Phi(\xi)}, \tag{29} \]

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Boundedness of Solutions

- Note that equation (26) can be rewritten as

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- When \( \Phi \) is a quadratic or cubic function or special function of \( U(\xi) \), we can analyze equation (29) qualitatively and numerically with classifications. For instance, if \( \Phi \) is quadratic, we take \( v_0 = \frac{b}{2a} \) and choose \( c = \frac{-2a}{\lambda\sqrt{b^2 - 4ad}} \) sufficiently small, we can obtain the solution of the type

\[
u(x, y, t) = \frac{3\beta^2 + \gamma + c}{25\alpha s} \text{sech}^2 \xi - \frac{6\beta^2 + \gamma + c}{25\alpha s} \tanh \xi \pm \frac{6\beta^2}{25\alpha s} + C_0.
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Boundedness of Solutions

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- When \( \Phi \) is a function with the lower and upper bounds, we can also find bounds of solutions of equation (29) by the comparison principle, which match well with the phase analysis described in [7].
In this talk, we provided a connection between the Abel equation of the first kind, an ordinary differential equation that is cubic in the unknown function, and the Korteweg-de Vries-Burgers equation, a partial differential equation that describes the propagation of waves on liquid-filled elastic tubes. We presented an integral form of the Abel equation with the initial condition.
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By virtue of the integral form and the Banach Contraction Mapping Principle we derived the asymptotic expansion of bounded solutions in the Banach space, and used the asymptotic formula to construct approximate solutions to the Korteweg-de Vries-Burgers equation.
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As an example, we presented the asymptotic behavior of traveling wave solution for a 2D KdV-Burgers equation which agrees well with existing results in the literature.
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As an example, we presented the asymptotic behavior of traveling wave solution for a 2D KdV-Burgers equation which agrees well with existing results in the literature.

Under certain conditions, we can also study bounds of traveling wave solutions of KdV-Burgers type equations by the comparison principle.
I would like to thank Xiaoqian Gong for discussions and help on computations.
Thank you.

KdV-Burgers Equation
Z. Feng
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