On the cohomology ring of compact hyperkähler manifolds

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1 Introduction and Motivation

The Chow ring of a smooth algebraic variety V, denoted $\mathrm{CH}^*(V)$, is an analogue of the cohomology ring that is more closely related to the algebraic, rather than topological, aspects of the variety. For a d-dimensional abelian variety A over a field k, let $\hat{A} = \mathrm{Pic}^0(A)$ be its dual, the variety parameterising principal line bundles on A, and for $a \in \hat{A}$ denote the line bundle parameterised by a as $\mathcal{L}_a$. The Poincaré line bundle $\mathcal{L}$ on $A \times \hat{A}$ is a line bundle satisfying the universal property such that $\forall a \in \hat{A}, \mathcal{L}_a \cong \mathcal{L}|_{A\times\{a\}}$ and $\mathcal{L}|_{\{0\}\times\hat{A}}$ is trivial. The first Chern class of $\mathcal{L}$ gives an element of $\mathrm{CH}^1(A \times \hat{A})$, which we denote L, that allows us to define the Fourier transform on the Chow group by:

$$\mathcal{F}(\alpha) = p_{2*}(e^L \cdot p_1^*(\alpha)) \quad \forall \alpha \in \mathrm{CH}^i(A),$$

where p_i is the projection of $A \times \hat{A}$ onto the ith factor, and $e^L = [A \times \hat{A}] + L + \frac{L^2}{2} + \cdots + \frac{L^{2d}}{d!}$. In other words, $\mathcal{F}$ is the usual map from $\mathrm{CH}(A)$ to $\mathrm{CH}(\hat{A})$ induced by e^L as an element of $\mathrm{CH}^*(A \times \hat{A})$. The following theorem was proved by Arnaud Beauville in [1]:

Theorem. *The Fourier transform on an abelian variety A of dimension d induces a canonical splitting*

$$\mathrm{CH}^i(A) = \bigoplus_{s=i-d}^{i} \mathrm{CH}^i(A)_s, \ \textit{where} \quad \mathrm{CH}^i(A)_s = \mathrm{CH}^i(A) \cap \mathcal{F}^{-1}(\mathrm{CH}^{d-i+s}(\hat{A})),$$

satisfying the following two properties:
a) $\mathrm{CH}^i(A)_s = \{\alpha \in \mathrm{CH}^i(A) : [n]^*\alpha = n^{2i-s}\alpha\}$ *for* $[n]$ *the multiplication by* n *map on* A
b) $\mathrm{CH}^i(A)_s \cdot \mathrm{CH}^j(A)_r \subset \mathrm{CH}^{i+j}(A)_{r+s}$

Recent work by Mingming Shen and Charles Vial in [5] has shown that a similar Fourier decomposition can be obtained for the Chow groups of certain hyperkähler varieties of $K3^{[2]}$ type, and provides evidence that a decomposition

should exist more generally. The aim of this project was to attempt to develop some of the results of this paper, and to understand some of the background material behind it.

I was interested in undertaking the project on a personal level, since it provided me with an opportunity to learn more about cohomology, the Chow ring and Kähler manifolds, subjects that I am very interested in learning more about and pursuing in the future. I was also able to see the wide variety of subjects and techniques that are used within algebraic geometry, and gain more of an appreciation for how diverse of a subject it is. The project also allowed me to do some hands on computations and to interact with the objects considered, without requiring me to have an extensive background in the material.

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2 Some facts about the cohomology and Chow rings

Throughout this write up I will make use of several results about cohomology, which I will summarise here.

Theorem (The cup product)**.** *If X is a topological space and R is a ring, given cochains α and β of degree p and q respectively, then we can define a degree $p+q$ cochain $\alpha \cup \beta$ by $\alpha \cup \beta(\sigma) = \alpha(\sigma \circ \iota_{0,\ldots,p})\beta(\sigma \circ \iota_{p,\ldots,p+q})$ for any singular simplex σ. Here $\iota_{i_1,\ldots,i_k}$ is the inclusion map of the simplex spanned by the vertices $\{i_1, \ldots, i_k\}$ into the standard $p+q$ simplex. This descends to an associative, distributive and functorial map $\mathrm{H}^p(X,R) \times \mathrm{H}^q(X,R) \to \mathrm{H}^{p+q}(X,R)$, giving $\mathrm{H}^*(X,R)$ the structure of a graded ring. It is not, however, commutative since we have $\alpha \cup \beta = (-1)^{p+q}\beta \cup \alpha$.*

Theorem (The Künneth decomposition)**.** *Let X and Y be topological spaces, p_i the projection maps from $X \times Y$ to the ith factor and $\mathbb{F}$ a field. Then the cross product map below is an isomorphism:*

$$\bigoplus_{i+j=k} \mathrm{H}^i(X,\mathbb{F}) \otimes \mathrm{H}^j(Y,\mathbb{F}) \to \mathrm{H}^k(X \times Y, \mathbb{F}), \quad \alpha \otimes \beta \mapsto \alpha \times \beta := p_1^*(\alpha) \cup p_2^*(\beta).$$

Theorem (Poincaré Duality)**.** *If M is a closed, oriented n-dimensional real manifold and $\mathbb{F}$ is a field, then the map below is an isomorphism:*

$$\mathrm{H}^k(M,\mathbb{F}) \to \mathrm{H}^{n-k}(M,\mathbb{F})^\vee, \quad x \mapsto (y \mapsto \textstyle\int_M x \cup y).$$

Theorem (Maps associated to elements of $\mathrm{H}^*(X \times Y)$)**.** *Given two manifolds as in the statement of Poincaré duality X and Y, any element α of $\mathrm{H}^*(X \times Y)$ induces a map $\alpha_* : \mathrm{H}^*(X) \to \mathrm{H}^*(Y)$ given by:*

$$\alpha_*(\sigma) = p_{2*}(\alpha \cup p_1^*(\sigma)) \ \ \forall \sigma \in \mathrm{H}^*(X),$$

where p_{2*} is the Poincaré dual of p_2^*. Moreover, if Z is a third manifold, and we take, for example, $p_{X,Y} : X \times Y \times Z \to X \times Y$ to be the projection map, then we may define a composition product $\circ$ combining elements of $\mathrm{H}^*(X \times Y)$ and $\mathrm{H}^*(Y \times Z)$ to give an element in $\mathrm{H}^*(X \times Z)$. Given $\alpha \in \mathrm{H}^*(X \times Y)$ and $\beta \in \mathrm{H}^*(Y \times Z)$:

$$\alpha \circ \beta = p_{X,Z*}(p_{X,Y}^*(\alpha) \cup p_{Y,Z}^*(\beta)).$$

This operation is functorial, in the sense that $(\alpha \circ \beta)_* = \alpha_* \circ \beta_*$.

For the sake of later computations it is worth looking at the exact form of the maps p_1^* and p_{2*} in terms of the Künneth decomposition, where X and Y are manifolds as in the statement of Poincaré duality, and $\mathbb{F}$ is a field. (This will be applicable for our purposes since a complex algebraic variety of dimension d defines a closed, oriented real manifold of dimension $2d$.) Then $\mathrm{H}^{\dim(X)}(X,\mathbb{F})$ is one dimensional, generated by v, say. For $\alpha \in \mathrm{H}^{\dim(X)}(X,\mathbb{F})$, write $\alpha = \overline{\alpha}v$.

Then, given $\alpha \in \mathrm{H}^k(X,\mathbb{F}), p_1^*(\alpha)$ is the element of $\mathrm{H}^k(X \times Y,\mathbb{F})$ corresponding to $\alpha \otimes [Y]$ under the Künneth decomposition. Additionally, given $\sigma = \sum_{i+j=k} \alpha_i \otimes \beta_j \in \mathrm{H}^k(X \times Y, F)$ then $p_{2*}(\sigma) = \overline{\alpha}_{\dim(X)}\beta_{k-\dim(X)}$ provided $\dim(X) \leq k$ and 0 otherwise. This allowed me to carry out the calculations in section 4.

Theorem (The Chow ring and map into cohomology)**.** *For a complex algebraic variety X, the Chow group $\mathrm{CH}^*(X)$ is roughly defined to be $\bigoplus_{i=0}^{\dim(X)} A^i(X)$ where $A^i(X)$ is the free group generated by all closed, irreducible subvarieties of X of codimension i, modulo rational equivalence (an equivalence relation on the set of varieties of the same dimension). This forms a ring when X is non-singular. To take the ring product of two classes $[Y]$ and $[Z]$ of codimension p and q respectively, requires a result known as "Chow's moving lemma". This says that when X is non-singular, we can find a cycle Y' rationally equivalent to Y that intersects Z in the correct dimension. Then $[Y] \cdot [Z]$ is (roughly) given by the sum of the classes of the irreducible components of $Y' \cap Z$, counted with multiplicity. In particular this produces a sum of cycles of codimension $p + q$. There is also a natural homomorphism $cl : \mathrm{CH}^*(X) \to \mathrm{H}^{2*}(X,\mathbb{Z})$ that takes a subvariety $[Y]$ to the Poincaré dual of the homology class given by Y.*

The cycle class map cl is still not-well understood in general. Indeed, the famed Hodge conjecture is a statement about the image of cl when X is a non-singular complex projective manifold. The kernel of cl is also poorly understood, though in general it is very large. This makes it harder to understand $\mathrm{CH}^*(X)$ than

$\mathrm{H}^*(X)$. In later sections, when we talk about a cycle in the Chow group corresponding to an element in cohomology, we mean a cycle that maps to that element under cl.

3 The Beauville-Bogomolov form and associated cohomology class

A complex manifold F is called Kähler if it can be equipped with a Hermitian metric whose fundamental form ω is closed, i.e. $d\omega = 0$. This is referred to as the Kähler metric. A hyperkähler manifold is a simply connected Kähler manifold such that $\mathrm{H}^0(F, \Omega_F^2)$ is spanned by a nowhere vanishing holomorphic two-form. Equivalently, its tangent bundle can be equipped with a quaternionic structure. Thus a hyperkähler manifold always has even complex dimension (and so real dimension a multiple of four). A $K3$ surface is another name for a compact hyperkähler manifold of dimension 2, and a hyperkähler variety is said to be of $K3^{[n]}$ type if it is deformation equivalent to the Hilbert scheme of length-n subschemes on a $K3$ surface.

If F is a compact hyperkähler manifold of dimension $2n$, then we have the following theorem:

Theorem (The Beauville-Bogomolov form; see [3])**.** *Let F be a compact hyperkähler manifold of dimension $2n$. Then $\mathrm{H}^2(F, \mathbb{Q})$ can be endowed with a canonical, non-degenerate symmetric bilinear form q_F satisfying the Fujiki relation:*

$$\int_F \alpha^{2n} = \frac{(2n)!}{2^n n!} c_F q_F(\alpha, \alpha)^n \quad \forall \alpha \in \mathrm{H}^2(F, \mathbb{Z}).$$

Here c_F is a positive rational number, q_F is known as the Beauville-Bogomolov form and $\frac{(2n)!}{2^n n!} c_F$ is the Fujiki constant. Moreover, the above relation can be rewritten as:

$$\int_F \alpha_1 \dots \alpha_{2n} = c_F \sum_{\sigma \in R} q_F(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \dots q_F(\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)}),$$

where R is a complete set of left coset representatives of the subgroup of the symmetric group S_{2n} generated by all transpositions of the form $(2j, 2j+1)$ and $(2j, 2k)(2j+1, 2k+1)$. In other words, each element of R represents a unique way of pairing up the $2n$ elements. This is much more useful when it comes to doing computations.

Now we show how this defines an element in the cohomology ring $\mathrm{H}^*(F \times F, \mathbb{Q})$, as in [5]. The symmetric bilinear forms on a finite dimensional vector space V correspond to the elements of $\mathrm{Sym}^2(V^\vee)$, and the non-degenerate ones induce an isomorphism from V to its dual, $V^\vee$. Through this we can associate a symmetric bilinear form q with an element in $\mathrm{Sym}^2(V)$, denoted q^{-1}. Explicitly,

if $e_1, \ldots, e_n$ is a basis of V and Q is the inverse of the matrix of the bilinear form q, then $q^{-1} = \sum q_{i,j} e_i \otimes e_j$. This allows us to associate to the Beauville-Bogomolov form q_F an element $q_F^{-1} \in \mathrm{H}^2(F, \mathbb{Q}) \otimes \mathrm{H}^2(F, \mathbb{Q})$. The Beauville-Bogomolov class $\mathfrak{B}$ is then defined to be the image of $0 \oplus q_F^{-1} \oplus 0$ in $\mathrm{H}^4(F \times F, \mathbb{Q})$ under the Künneth decomposition of cohomology groups. Note that since F is simply connected, by Poincaré duality $\mathrm{H}^i(F, \mathbb{Q}) = 0$ for $i = 1, 3$, so we may decompose $\mathrm{H}^4(F \times F, \mathbb{Q})$ as a sum of three, rather than five terms.

We also define b_i and b using p_i the projection maps on $F \times F$ and $i_\Delta : F \to F \times F$ the diagonal embedding map as follows:

$$b := i_\Delta^*(\mathfrak{B}), \quad b_i := p_i^*(b)$$

If we work with cohomology in complex coefficients, we can extend q_F to $\mathrm{H}^2(F, \mathbb{C})$, and since it remains non-degenerate, we can pick an orthonormal basis $e_1, \ldots, e_r$ with respect fo q_F. Then $\mathfrak{B} = \sum e_i \otimes e_i, b = \sum e_i^2, b_1 = \sum e_i^2 \otimes 1$ and $b_2 = \sum 1 \otimes e_i^2$. This simplification makes performing computations much easier.

4 Attempting to develop the methods of [5]

4.1 Continuing with the same approach

The work in [5] begins by defining the cohomological Fourier transform on a compact hyperkähler manifold of complex dimension $2n$ analogously to the Fourier transform on the Chow ring, but using the Beauville-Bogomolov class $\mathfrak{B}$ rather than the Poincaré line bundle L as follows:

$$[\mathcal{F}](\alpha) = p_{2*}(e^{\mathfrak{B}} \cup p_1^* \alpha) \quad \forall \alpha \in \mathrm{H}^*(F, \mathbb{Q}).$$

After investigating the nature of the cohomological Fourier transform and showing that it produces a decomposition of the cohomology ring, key relations satisfied by $\mathfrak{B}$ were identified, and it was proved that finding a cycle in the Chow group corresponding to $\mathfrak{B}$ that also satisfied analogous relations would produce a Fourier decomposition on the Chow groups of the desired form, for varieties of $K3^{[2]}$ type. It was then shown that such a cycle exists for $S^{[2]}$ the Hilbert scheme of length 2 subschemes of S, for S a $K3$ surface and for the variety of lines on cubic fourfolds. It was further shown that in these cases, the decomposition is compatible with the ring structure on the Chow group.

I began to investigate the behavior of the action of degree n monomials (with the cup product for multiplication) in $\mathfrak{B}, b_i$ on S^*, the subring of the cohomology ring $\mathrm{H}^*(F, \mathbb{C})$ generated by the degree two classes. Here F is any compact hyperkähler manifold of complex dimension $2n$, rather than just a variety of $K3^{[2]}$ type. In particular I looked for linear combinations of the monomials that would act as the projection maps from S^* to S^i for each i, denoted π_i. Summing all of the elements acting as projection maps would then give an

element of $\mathrm{H}^{2n}(F \times F, \mathbb{C}) \cong \mathrm{Hom}(\mathrm{H}^*(F, \mathbb{C}), \mathrm{H}^*(F, \mathbb{C}))$ that acts as the identity when restricted to S^*. It was a relation of this form that was used in the case of varieties of $K3^{[2]}$ type in [5].

It was also shown in [5], based on the work of Markman in [4] that for any hyperkähler variety F of $K3^{[n]}$ type, there is a cycle $L \in \mathrm{CH}^2(F \times F)$ whose cohomology class is $\mathfrak{B} \in \mathrm{H}^4(F \times F, \mathbb{Q})$.

Unfortunately, even for the case $n = 3$ (the simplest case not covered in [5]) this approach proved difficult. Calculations became much more involved, and though the following relations could be obtained:

$$\pi_0 = \frac{1}{c_F r(r+2)(r+4)} b_1^3 \qquad \pi_1 = \frac{1}{c_F (r+2)(r+4)} b_1^2 \mathfrak{B}$$

$$\pi_2 = \frac{1}{2c_F(r+4)}[b_1\mathfrak{B}^2 - \frac{1}{(r+2)} b_1^2 b_2]$$

$$\pi_4 = \frac{1}{2c_F(r+4)}[b_2\mathfrak{B}^2 - \frac{1}{(r+2)} b_1 b_2^2]$$

$$\pi_5 = \frac{1}{c_F (r+2)(r+4)} b_2^2 \mathfrak{B} \qquad \pi_6 = \frac{1}{c_F r(r+2)(r+4)} b_2^3$$

it turns out that π_3 is not a linear combination of $(\mathfrak{B}^3)$ and $(b_1 b_2 \mathfrak{B})$ (the monomials that act non-trivially on S^3). We can still produce a linear combinations of elements of $\mathrm{H}^*(F \times F)$ that acts as π_3 provided we also use the composition product of elements, as described in the previous section. We find that:

$$\pi_3 = \frac{7}{36c_F}\mathfrak{B}^3 - \frac{r+16}{24c_F(r+4)} b_1 b_2 \mathfrak{B} + \frac{1}{72c_F^2(r+4)}[\mathfrak{B}^3 \circ b_1 b_2 \mathfrak{B} + b_1 b_2 \mathfrak{B} \circ \mathfrak{B}^3] - \frac{1}{216c_F^2}\mathfrak{B}^3 \circ \mathfrak{B}^3.$$

Unfortunately, this does not suggest a pattern that could be followed for higher dimensions, so a different approach was needed.

4.2 Trying a new approach

Rather than finding a combination of monomials of the form $b_1^{i_1} b_2^{i_2} \mathfrak{B}^{i_3}$ acting as the projection maps, we can instead examine the minimal and characteristic polynomials of the maps $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_* : S^k \to S^k$. Provided the constant terms are non-zero, we can write the identity map on S^k as a linear combination of powers of $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$. Since $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$ acts as 0 on $\mathrm{H}^j(F, \mathbb{C})$ for $j \neq 2k$ (by the observations in section 2), this relationship also gives an expression for the projection maps. Explicitly, if $p(X) = \sum_{i=0}^m a_i X^i$ is the minimal polynomial of $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$ on S^k, then $\pi_k : S^* \to S^k$ is given by $\frac{-1}{a_0} \sum_{i=1}^m a_i (\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})^i$.

The constant term of the minimal and characteristic polynomials are products (with multiplicities) of the eigenvalues of the map, so this approach will work provided that $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$ is an isomorphism on S^k. This fact is proved in 4.5 below, but first I provide some necessary preliminary results.

4.3 A selection of useful results

In [2], Fedor Bogomolov proves several results about compact hyperkähler manifolds F of dimension $2n$ which are important for our purposes. In particular, theorem 1 below is essential for section 4.5. They were originally proved by Misha Verbitsky, but the approach given by Bogomolov is more elementary. Bogomolov begins with the following theorem:

Theorem. *Define $Q \subset \mathrm{H}^2(F, \mathbb{C})$ as the zero set of the equation $\alpha^{n+1} = 0$. Then this is a smooth quadric and the Beauville-Bogomolov form q_F is defined to be the quadratic form defining Q, normalised so that $q_F(l) = 1$ for the class l of a Kähler metric with $l^{2n} = 1$.*

The q_F used throughout this write up is actually the symmetric bilinear form associated to the above quadratic one, and we write q for q_F^{-1}. Bogomolov then goes on to show that for $\alpha \in \mathrm{H}^2(F, \mathbb{C}) \setminus Q, \alpha^{2n} \neq 0$ and using this he then derived the following two theorems:

Theorem 1. *Write S^1 for $\mathrm{H}^2(F, \mathbb{C})$ and let S^* be the subring of $\mathrm{H}^*(F, \mathbb{C})$ generated by S^1. Then S^{2n} is one dimensional generated by q^n and the product map $S^i \times S^{2n-i} \to S^{2n}$ is a dual pairing. In particular $S^i \cong (S^{2n-i})^\vee$.*

Theorem 2. *S^* is isomorphic to the quotient of $\mathrm{Sym}^*(S^1)$ by a homogeneous ideal J' such that the graded components of every element of J' have degree at least $n + 1$.*

Together these give that for $k \leq n$, $S^k \cong \mathrm{Sym}^k(S^1)$ and $S^{2n-k} \cong (S^k)^\vee$ in a natural way.

It is worth noting that Q being a smooth quadric is a very strong result, and originally led Bogomolov to publish a paper claiming that no hyperkähler manifolds existed in dimension larger than 2. It was almost 4 years before Fujiki constructed a counterexample. Even so, q_F as a symmetric bilinear form has received a lot of use in hyperkähler geometry and allowed much progess to be made.

To prove the above theorems, we first need to introduce the second order Laplacian differential operator Δ corresponding to q. Since we have picked a basis $\{e_1, \ldots, e_r\}$ of S^1 orthonormal with respect to the bilinear form q_F, $\Delta = \frac{1}{2r} \sum_{i=1}^{r} \frac{\partial^2}{\partial e_i^2}$. This then acts on $\mathrm{Sym}^*(S^1)$ viewed as the ring of symmetric polynomials in the e_i.

We then write $O(q_F)$ for the orthogonal group with respect to q_F, and notice that Δ commutes with the action of $O(q_F)$ on $\mathrm{Sym}^*(S^1)$.

Finally, we denote by I_k the subspace of harmonic elements in $\mathrm{Sym}^k(S^1)$. The monomials x^k for $x \in Q$ are contained in I_k, since for $x \in S^1$:

$$\Delta(x^k) = k(k-1)x^{k-2} q_F(x, x).$$

We list the following collection of well known results about harmonic polynomials to use throughout this section:

Lemma. *1)As an $O(q_F)$ module, $\mathrm{Sym}^n(S^1)$ is a direct sum of the non isomorphic modules $q^r I_{n-2r}$.*

2)The product map takes $I_r \times I_1$ surjectively onto $I_{r+1} + qI_{r-1}$

3) $\Delta : \mathrm{Sym}^n(S^1)/I_n \to \mathrm{Sym}^{n-2}(S^1)$ is an isomorphism

4) I_k is generated by the monomials x^k for $x \in Q$

And note that from part 2), we may deduce by induction:

5)*$I_r \times I_k$ maps surjectively onto $\sum_{l=0}^{k} q^l I_{r+k-2l}$, where $k \leq r$.*

Let J be the ideal generated by I_{n+1} and let R^* be the quotient $\mathrm{Sym}^*(S^1)/J$. We first show that R^{2n} is one dimensional generated by the class of q^n and that the product map on $R^i \times R^{2n-i} \to R^{2n}$ is a dual pairing. We then show that $R^* \cong S^*$ which proves theorems 1 and 2.

Proof : Using parts 3 and 5 of the lemma, it follows that $J^{n+k} = \sum_{0 \leq l < k} q^l I_{n+k-2l}$ and then that $R^{n+k} = \sum_{k \leq l \leq \frac{n+k}{2}} q^l I_{n+k-2l}$. In particular, $R^{2n} = q^n I_0$.

To show that the product map is a dual pairing, first take any $x \in I_s$ and notice that since I_s is an irreducible $O(q_F)$-invariant subspace, by part 5 of the lemma, it must be that the $O(q_F)$ subspace generated by xI_s is $\mathrm{Sym}^{2s}(S^1)$. Thus the $O(q_F)$-invariant projection map from $\mathrm{Sym}^{2s}(S^1) \to I_0 q^s$ is non-trivial when restricted to xI_s.

Now suppose we have $y \in R^t$, written as $y = \sum q^s y_s$ where $y_s \in I_{t-2s}$. Taking r to be minimal such that $y_r \neq 0$, there is some $x \in I_{t-2r}$ such that the projection of xy_r to $q^{t-2r} I_0$ is non-zero. Then using the product formula gives that $q^{n-t+r} x \in R^{2n-t}$ is such that $q^{n-t+r} xy$ projects non-trivially onto $q^n I_0$ and so is non-zero in R^{2n}. Thus the product map is a dual pairing as required.

Finally we show that $R^* = S^*$, which proves both of the theorems at once:

Proof : First we show that J' contains J, and then show that the map $\varphi : \mathrm{Sym}^*(S^1)/J = R^* \to S^*$ is injective, since it is certainly surjective. Since J is generated by I_{n+1} it suffices to show that $I_{n+1} \subset J'$. But this is immediate from part 4 of the lemma and the fact that Q is exactly the set of points $x \in S^1$ such that $x^{n+1} = 0$. Finally, notice that since R^* is self-dual and R^{2n} is one-dimensional, any non-zero ideal of R^* contains R^{2n}. But clearly the restriction of φ to R^{2n} is injective, and so the map has trivial kernel, and we are done.

4.4 $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$ is an isomorphism on S^k

Here we prove the result mentioned in section 4.2. Recall that we have maps $\varphi_k : \mathrm{Sym}^k(S^1) \to \mathrm{H}^{2k}(F, \mathbb{C})$, descending from the map on the kth tensor power of S^1 that sends $e_{i_1} \otimes \cdots \otimes e_{i_k}$ to $e_{i_1} \ldots e_{i_k}$, and the image of φ_k is S^k.

As a result of the theorems in the previous section, for $k \leq n$, $U_k = \{e(I) := e_1^{i_1} \ldots e_r^{i_r} : \sum i_j = k\}$ is a basis of $S^k \cong \mathrm{Sym}^k(S^1)$ and the product map on $S^k \times S^{2n-k}$ is a duality pairing, so that $S^{2n-k} = (S^k)^\vee$ and $U_k^\vee$ is a basis of S^{2n-k}.

We now show that $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_* : S^k \to S^k$ is an isomorphism by showing that $(\mathfrak{B}^k)_* : S^{2n-k} \to S^k$ and $(\mathfrak{B}^{2n-k})_* : S^k \to S^{2n-k}$ are isomorphisms for all $k \leq n$.

If $I = (i_1, \ldots, i_r)$ is a partition of k into r parts, then we denote by $\binom{k}{I}$ the

multinomial coefficient $\binom{k}{i_1,\ldots,i_r}$. Then $\mathfrak{B}^k = \sum \binom{k}{I} e(I) \otimes e(I)$, and by rescaling this basis to $e'(I) = \binom{k}{I}^{\frac{1}{2}} e(I)$, we get $\mathfrak{B}^k = \sum e'(I) \otimes e'(I)$. Thus $\mathfrak{B}^k_*(e'(I)^\vee) = e'(I)$, so $\mathfrak{B}^k$ is an isomorphism.

$\mathfrak{B}^{2n-k}_*(e'(A)) = \sum(\int e'(A)e'(I))e'(I)$, where we sum over all possible I, the partitions of $2n-k$ into r parts. Moreover, since $S^{2n-k} = (S^k)^\vee, e'(I) = \sum(\int e'(J)e'(I))e'(J)^\vee$, where now we sum over all partitions J of k into r parts. Thus the matrix of $\mathfrak{B}^{2n-k}_*$ with respect to bases $e'(A), (e'(A))^\vee$ is $M = NN^T$, where $N_{A,I} = \int e'(A)e'(I)$ for A a partition of k and I a partition of $2n-k$. N gives a linear map from $\mathrm{Sym}^{2n-k}(S^1)$ to S^k and since N is a real matrix, M is an isomorphism if and only if N is surjective. This is because N and N^T have the same rank, so N being surjective is equivalent to N^T being injective. Clearly M being an isomorphism implies that N^T is injective, and the converse can be shown as follows: Suppose $v \in \mathbb{R}^n$ is such that $Mv = 0$. Then $NN^Tv = 0$ so $v^TNN^Tv = 0$. In other words, $(N^Tv) \cdot (N^Tv) = 0$, and so $N^Tv = 0$ also. Then by injectivity, v is 0, so M is injective, whence an isomorphism.

But it is clear that $N = \mathfrak{B}^k_* \circ \varphi_{2n-k}$ and so is surjective, giving that $\mathfrak{B}^{2n-k}_*$ is an isomorphism as required. $\square$

4.5 A decomposition of cohomology

So far we have been working with cohomology in complex coefficients, since this is how the results from [2] were framed. However, we are ultimately interested in using rational coefficients. We therefore define $\mathbf{S}^*$ similarly to S^* to be the subring of $\mathrm{H}^*(F, \mathbb{Q})$ generated by the degree 2 classes. The fact that $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$ restricts to an isomorphism on S^k implies that it restricts to an isomorphism on $\mathbf{S}^k$, in the same way that if an $n \times n$ matrix with rational coefficients gives an isomorphism on $\mathbb{C}^n$ then it also gives an isomorphism on $\mathbb{Q}^n$.

Notice that since $\mathfrak{B}^k$ is an element of $\mathrm{H}^{2k}(F, \mathbb{Q}) \otimes \mathrm{H}^{2k}(F, \mathbb{Q})$, written as $\sum a_i \otimes b_i$, then for any element $\alpha \in \mathrm{H}^{4n-2k}(F, \mathbb{C})$, $\mathfrak{B}^k_*(\alpha) = \sum(\int a_i\alpha)b_i$, so the image of $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$ on $\mathrm{H}^{2k}(F, \mathbb{Q})$ is contained in $\mathbf{S}^k$. But we already know that $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$ acts isomorphically on $\mathbf{S}^k$, so we may decompose $\mathrm{H}^{2k}(F, \mathbb{Q})$ as the direct sum $\mathbf{S}^k \oplus T^k$ where T^k is the kernel of $\mathfrak{B}^{2n-k}_*$ acting on $\mathrm{H}^{2k}(F, \mathbb{Q})$. It is then immediate that the element of $\mathrm{H}^{4n}(F \times F, \mathbb{Q})$ 4.3 obtained from the minimal polynomial of $(\mathfrak{B}^k \circ \mathfrak{B}^{2n-k})_*$ acting as the identity on $\mathbf{S}^k$ (as in 4.3) acts as the projection map from $\mathrm{H}^{2k}(F, \mathbb{Q})$ onto $\mathbf{S}^k$ with respect to the decomposition given above.

For projective hyperkähler manifolds of $K3^{[n]}$ type, we know that we can lift the element $\mathfrak{B}$ of cohomology to the Chow ring, as mentioned in 4.1. In other words, $\mathfrak{B}$ is the class corresponding to an algebraic cycle $L \in \mathrm{CH}^2(F \times F)$. We can use this to lift the elements $\mathfrak{B}^k \circ \mathfrak{B}^{2n-k}$, and this fact together with the observations made in 4.2 immediately implies the following proposition:

Proposition. *Let F be a projective hyperkähler manifold of $K3^{[n]}$ type. Then there exists a cycle $\pi \in \mathrm{CH}^{2n}(F \times F)$ such that $[\pi] \in \mathrm{H}^{4n}(F \times F, \mathbb{Q}) \cong \mathrm{Hom}(H^*(F, \mathbb{Q}), H^*(F, \mathbb{Q}))$ is the projection map on $\boldsymbol{S}^*$. i.e. for each k, $[\pi]$ restricted to $H^{2k}(F, \mathbb{Q})$ is the projection map onto $\boldsymbol{S}^k$ with respect to the decomposition $H^k(F, \mathbb{Q}) = \boldsymbol{S}^k \oplus T^k$ and $[\pi]$ restricted to $H^{2k+1}(F, \mathbb{Q})$ is the zero*

map.

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