Calculus Refresher for M408D – Day 2 – Solutions

1. Find the intervals on which the function f increases, and the intervals on which f decreases.

$$f(x) = x\ln x.$$

The intervals on which a function increases or decreases are divided by places where the derivative either equals zero or is undefined.

$$f(x) = x\ln x$$

$$f'(x) = (1)\ln x + x\left(\frac{1}{x}\right)$$

$$f'(x) = \ln x + 1$$

$\underline{f'(x) = 0}$

$$\ln x + 1 = 0$$

$$\ln x = -1$$

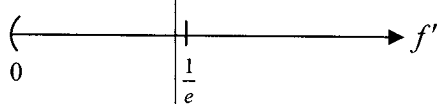
$$e^{\ln x} = e^{-1}$$

$$x = \frac{1}{e}$$

$\underline{f'(x)\text{ undefined}}$

$f'(x)$ is undefined for $x \le 0$
However, this interval is outside the domain of the original function.
Therefore, $f'(x)$ is defined everywhere on the domain of $f(x)$, that is, $x > 0$

Placing this value on a numberline, in order to use the first derivative test…

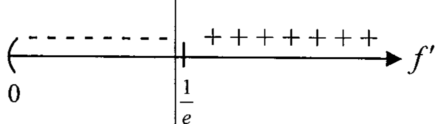


Testing values in each interval into the derivative:

$$f'\left(\frac{1}{e^2}\right) = \ln\left(e^{-2}\right) + 1 = -2 + 1 = -1$$

$$f'(e) = \ln(e) + 1 = 1 + 1 = 2$$

This gives the first derivative sign chart:



Therefore $f(x) = x\ln x$ is decreasing on the interval $\left(0, \frac{1}{e}\right)$ and increasing on the interval $\left(\frac{1}{e}, \infty\right)$

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2. Find the absolute Maximum and Minimum of

$$f(x) = x^4 - 2x^2 + 3 \text{ on } [-2,3]$$

A continuous function on a closed interval will achieve its absolute maximum and absolute minimum values at either critical points (places where $f' = 0$ or f' is undefined) or the endpoints of the interval.
Therefore, we want to find the function's height at these values.

$$f(x) = x^4 - 2x^2 + 3$$
$$f'(x) = 4x^3 - 4x$$

$\underline{f'(x) = 0}$

$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$4x = 0 \quad ; \quad x^2 - 1 = 0$$
$$x = 0 \quad ; \quad x = \pm 1$$

$\underline{f'(x) \text{ undefined}}$

Polynomials are defined everywhere on $\mathbb{R}$
That is, $4x^3 - 4x$ is not undefined for any value x

So, our critical points are $x = -1, 0, 1$ and interval endpoints are $x = -2, 3$
Testing the function's height at all these values, we have...

$$f(x) = x^4 - 2x^2 + 3$$
$$f(-2) = (-2)^4 - 2(-2)^2 + 3 = 11$$
$$f(-1) = (-1)^4 - 2(-1)^2 + 3 = 2$$
$$f(0) = (0)^4 - 2(0)^2 + 3 = 3$$
$$f(1) = (1)^4 - 2(1)^2 + 3 = 2$$
$$f(3) = (3)^4 - 2(3)^2 + 3 = 66$$

And so, on the interval $[-2,3]$, f has an absolute maximum value at $x = 3$ of 66 and an absolute minimum value at $x = -1$ and $x = 1$ of height 2

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3. A right circular cylinder (with a bottom and a top) is to be designed to hold 12 fluid ounces of a soft drink and to use a minimum of material in its construction. Find the required dimensions for the container. [1 fl.oz. $\approx$ 1.8 in^3]

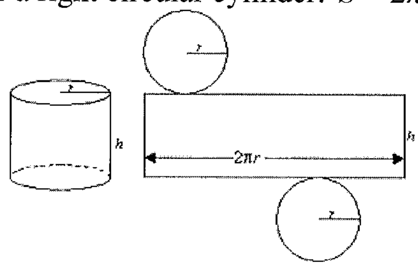
My preferred way of doing most optimization problems is to follow five stages:

1) Summarize the problem in two words (e.g. "maximize profit")
2) State the general equation for the function to optimize (e.g. profit = revenue – cost)
3) Determine the constraint in the problem statement, something that relates the known and unknown variables
4) Determine a specific equation for the function to optimize (usually by solving the constraint equation for one variable and substituting into the general equation)

5) Determine the function's absolute maximum or minimum value using techniques learned prior

So, here goes…

1) Minimize Surface Area
2) Surface Area formula for a right circular cylinder: $S = 2\pi r^2 + 2\pi rh$



3) The can must hold 12 oz. or 21.6 in^3

$$V = \pi r^2 h$$

$$21.6 = \pi r^2 h$$

4) We need to create a surface area formula in one variable. Therefore, we solve the constraint $21.6 = \pi r^2 h$ for one variable and substitute into the surface area formula.

Since the h variable appears once in the S equation (and appears linearly in the constraint), it appears reasonable to solve for it, as opposed to solving for r

$$21.6 = \pi r^2 h$$

$$\frac{21.6}{\pi r^2} = h$$

And substituting into the surface area formula, we have…

$$S = 2\pi r^2 + 2\pi rh$$

$$S = 2\pi r^2 + 2\pi r\left(\frac{21.6}{\pi r^2}\right)$$

$$S = 2\pi r^2 + \frac{43.2}{r}$$

5) Finally we take a derivative, determine critical points, and finish the optimization process…

$$S(r) = 2\pi r^2 + \frac{43.2}{r}$$

$$S(r) = 2\pi r^2 + 43.2r^{-1}$$

$$S'(r) = 4\pi r - 43.2r^{-2}$$

$$S'(r) = 4\pi r - \frac{43.2}{r^2}$$

$\underline{S'(r)=0}$

$$4\pi r-\frac{43.2}{r^2}=0$$

$$4\pi r=\frac{43.2}{r^2}$$

$$r^3=\frac{10.8}{\pi}$$

$$r=\sqrt[3]{\frac{10.8}{\pi}}$$

$r\approx 1.5$ in

$\underline{S'(r)\text{ undefined}}$

$4\pi r-\dfrac{43.2}{r^2}$ is defined for all values of r except $r=0$

However, it's not possible to have a physical cylinder with radius zero.

So, $r=0$ is out of our natural domain.

And now to verify that a radius of 1.5 (approximately) will minimize the surface area, we wish to show that an absolute minimum occurs here.

In other optimization problems, when there is an obvious closed interval for a variable, I will often verify extreme values by plugging them into the function. In our current case, it is clear that $r>0$, however, there is not a clear upper bound on r

In these cases, I use the first derivative test…

(———— | ————→ S'

0 1.5

$$S'(r)=4\pi r-\frac{43.2}{r^2}$$

$$S'(1)=4\pi(1)-\frac{43.2}{(1)^2}<0$$

$$S'(2)=4\pi(2)-\frac{43.2}{(2)^2}>0$$

(- - - - - - - - | + + + + + + + + → S'

0 1.5

And so an absolute minimum occurs at $r=\sqrt[3]{\dfrac{10.8}{\pi}}$ inches

And using the constraint to determine the can's height…

$$h=\frac{21.6}{\pi r^2}$$

$$h=\frac{21.6}{\pi\left(\sqrt[3]{\frac{10.8}{\pi}}\right)^2}$$

$h\approx 3$ in

And so, a can holding 12 fluid ounces has minimal surface area when it has approximately the dimensions of 1.5 in radius and 3 in height.

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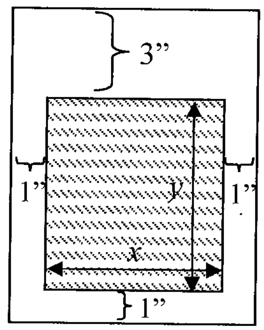
4. A poster is to have a total area of 72 in^2 with 1-inch margins at the bottom and sides and a 3-inch margin at the top. What dimensions will give the largest printed area?

We'll follow the same five stages as in problem 3…

1) Maximize Area (of the printed space)
2) Let x = width of the printed space and y = height of the printed space
   Then, the area of the printed space is given by: $A = xy$



3) The total area of the poster will be 72 in^2
   In terms of our stated variables, this means:
   $72 = (\text{width of poster}) \times (\text{height of poster})$
   $72 = (x+2)(y+4)$

4) We would like to rewrite our optimizing function $A = xy$ in terms of x or y only
   We could solve the constraint equation $72 = (x+2)(y+4)$ for either variable, and solving for either seems about as easy as the other, so we'll solve for y
   $$72 = (x+2)(y+4)$$
   $$\frac{72}{x+2} - 4 = y$$
   And substituting into our function $A = xy$, we have
   $$A = x\left(\frac{72}{x+2} - 4\right)$$
   $$A = x\left(\frac{72}{x+2} - \frac{4(x+2)}{(x+2)}\right)$$
   $$A = x\left(\frac{64-4x}{x+2}\right)$$
   $$A = \frac{64x-4x^2}{x+2}$$

5) Finally, optimizing the area function, we have

$$A(x)=\frac{64x-4x^2}{x+2}$$

$$A'(x)=\frac{(x+2)\left[64x-4x^2\right]'-\left(64x-4x^2\right)[x+2]'}{(x+2)^2}$$

$$A'(x)=\frac{(x+2)(64-8x)-\left(64x-4x^2\right)(1)}{(x+2)^2}$$

$$A'(x)=\frac{64x-8x^2+128-16x-64x+4x^2}{(x+2)^2}$$

$$A'(x)=\frac{-4x^2-16x+128}{(x+2)^2}$$

$$A'(x)=\frac{-4\left(x^2+4x-32\right)}{(x+2)^2}$$

$$A'(x)=\frac{-4(x+8)(x-4)}{(x+2)^2}$$

$\underline{A'(x)=0}$

$$\frac{-4(x+8)(x-4)}{(x+2)^2}=0$$

$$-4(x+8)(x-4)=0$$

$x+8=0$; $x-4=0$

$x=-8$; $x=4$

$\underline{A'(x)\text{ undefined}}$

$\frac{-4(x+8)(x-4)}{(x+2)^2}$ is undefined when

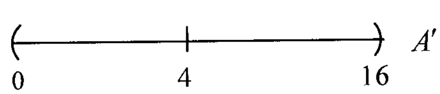
$$(x+2)^2=0$$

$$x=-2$$

Since x is a physical distance in our problem, we throw out the negative solutions of $x=-2$ and $x=-8$, and just consider the value $x=4$

In determining the range of values possible for x, consider that $x>0$ and when x is near the value 16, we would have a total width of 18 inches, and height of 4 inches, giving us the required 72 in^2

In verifying that a local maximum occurs at $x=4$, my preference is to always use the first derivative test.



$$A'(x)=\frac{-4(x+8)(x-4)}{(x+2)^2}$$

$$A'(1)=\frac{-4(1+8)(1-4)}{(1+2)^2}=\frac{(-)(+)(-)}{(+)}=+$$

$$A'(5)=\frac{-4(5+8)(5-4)}{(5+2)^2}=\frac{(-)(+)(+)}{(+)}=-$$

$$\overset{+++|+++|\ ---------}{\underset{0\qquad\qquad 4\qquad\qquad\qquad 16}{(\!\!\longrightarrow\!\!)}}\quad A'$$

Therefore, $A(x)$ increases to $x=4$ and decreases afterward, making $x=4$ an absolute maximum on the interval $(0,16)$

Finally, using the constraint $72=(x+2)(y+4)$, we determine the height of the printed area:

$$72=(x+2)(y+4)$$
$$72=((4)+2)(y+4)$$
$$y=8$$

And so, the area of the printed section of the poster is maximized when the printed area is 4 inches wide and 8 inches tall.

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5. Determine all real numbers x that make the second derivative of $y=\frac{1}{2}\tan x+\sin x$ equal zero or undefined.

$$y=\frac{1}{2}\tan x+\sin x$$
$$\frac{dy}{dx}=\frac{1}{2}\sec^2 x+\cos x$$
$$=\frac{1}{2}(\sec x)^2+\cos x$$
$$\frac{d^2y}{dx^2}=(\sec x)(\sec x\tan x)-\sin x$$
$$=\tan x\sec^2 x-\sin x$$

For the purposes of determining where $\frac{d^2y}{dx^2}$ is equal to zero or undefined, we'll want to modify the expression a bit.

$$\frac{d^2y}{dx^2}=\tan x\sec^2 x-\sin x$$
$$=\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}\cdot\frac{1}{\cos x}-\sin x$$
$$=\sin x\left(\frac{1}{\cos^3 x}-1\right)$$

$$\frac{d^2y}{dx^2}=0$$

$\frac{d^2y}{dx^2}$ undefined

$\sin x\left(\frac{1}{\cos^3 x}-1\right)$ is undefined when

$\sin x = 0$

$x = \cdots, -\pi, 0, \pi, 2\pi, 3\pi, \cdots$

or

$\dfrac{1}{\cos^3 x} - 1 = 0$

$\cos x = 1$

$x = \cdots, -2\pi, 0, 2\pi, 4\pi, 6\pi, \cdots$

Both of these cases are captured with

$x = k\pi \qquad k$ an integer $(k \in \mathbb{Z})$

$\cos x = 0$

$x = \cdots, -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2} \cdots$

These values can be written as

$x = \dfrac{(2k-1)\pi}{2} \qquad k$ an integer $(k \in \mathbb{Z})$

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6. Find all the values of x at which the graph of $y = x^2 + 4\sin x$ changes concavity on $[0, \pi]$

The first derivative is used to determine a function's slope, local max/min values and intervals of increase or decrease.
The second derivative is used to determine a function's intervals of concavity and points of inflection.
Intervals of concavity are divided by values where the second derivative is either zero or undefined.

$\dfrac{dy}{dx} = 2x + 4\cos x$

$\dfrac{d^2y}{dx^2} = 2 - 4\sin x$

$\dfrac{d^2y}{dx^2} = 0$

$2 - 4\sin x = 0$

$\sin x = \dfrac{1}{2}$

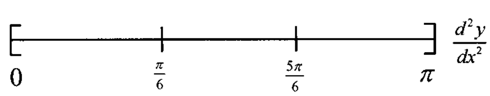
$x = \cdots, \dfrac{-7\pi}{6}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \cdots$

Solutions in $[0, \pi]$ are: $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\dfrac{d^2y}{dx^2}$ undefined

$2 - 4\sin x$ is defined for all real numbers x

Using a numberline to record the second derivative results…



Plugging in sample values…

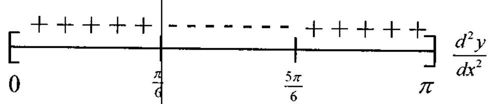
$$\frac{d^2y}{dx^2} = 2 - 4\sin x$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 - 4\sin(0) = 2$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{2}} = 2 - 4\sin\left(\frac{\pi}{2}\right) = -2$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\pi} = 2 - 4\sin(\pi) = 2$$

(We can use the endpoints of the intervals since they are not zeroes of the second derivative.)
And so our numberline would be:



Therefore, the function $y = x^2 + 4\sin x$ changes concavity at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

Furthermore, we could also state:

The graph of $y = x^2 + 4\sin x$ is concave upward on the interval $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right)$ and concave downward on the interval $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

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7. Consider the function $y = x^3 - 6x^2 + 9x$.

a. Determine the open intervals on which the function is increasing.

Using the first derivative test, we have…

$$y = x^3 - 6x^2 + 9x$$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$\frac{dy}{dx} = 0$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

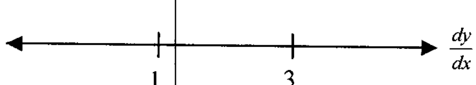
$x - 1 = 0 \quad ; \quad x - 3 = 0$

$x = 1 \quad ; \quad x = 3$

$\frac{dy}{dx}$ undefined

$3x^2 - 12x + 9$ is a polynomial and is therefore defined for all real numbers

Using a numberline to record the results…



Plugging in sample values…

$$\frac{dy}{dx}=3x^2-12x+9$$

$$\left.\frac{dy}{dx}\right|_{x=0}=3(0)^2-12(0)+9=9>0$$

$$\left.\frac{dy}{dx}\right|_{x=2}=3(2)^2-12(2)+9=-3<0$$

$$\left.\frac{dy}{dx}\right|_{x=4}=3(4)^2-12(4)+9=9>0$$

And so our numberline would look like…

$$\overset{+++++}{\longleftarrow\!\!-\!\!-}\!\!\underset{1}{|}\!\!\overset{------}{-\!\!-\!\!-}\!\!\underset{3}{|}\!\!\overset{+++++}{-\!\!-\!\!\longrightarrow}\ \frac{dy}{dx}$$

Therefore, the function $y=x^3-6x^2+9x$ is increasing on the interval $(-\infty,1)\cup(3,\infty)$ and decreasing on the interval $(1,3)$

b. Determine the relative extrema.

Using our results from part (a), and that the values 1 and 3 are in the domain of $y=x^3-6x^2+9x$, we have that y increases to $x=1$ and decreases after, causing a local maximum to occur at $x=1$, specifically at the point $(1,4)$

Similarly, y decreases to $x=3$ and increases after, causing a local minimum at $x=3$, specifically at the point $(3,0)$

c. Determine the intervals on which the function is concave up or down.

Using the second derivative test, we have…

$$\frac{dy}{dx}=3x^2-12x+9$$

$$\frac{d^2y}{dx^2}=6x-12$$

$\frac{d^2y}{dx^2}=0$	$\frac{d^2y}{dx^2}$ undefined
$6x-12=0$	$6x-12$ is a polynomial and is therefore defined for all real numbers
$x=2$	

Using a numberline to record the results…

$$\longleftarrow\!\!-\!\!-\!\!-\!\!\underset{2}{|}\!\!-\!\!-\!\!-\!\!\longrightarrow\ \frac{d^2y}{dx^2}$$

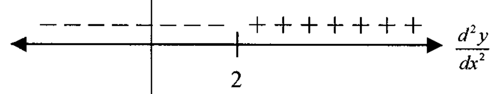
Plugging in sample values…

$$\frac{d^2y}{dx^2} = 6x - 12$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 6(0) - 12 = -12 < 0$$

$$\left.\frac{dy}{dx}\right|_{x=3} = 6(3) - 12 = 6 > 0$$

And so our numberline would look like…



Therefore, the function $y = x^3 - 6x^2 + 9x$ is concave downward on the interval $(-\infty, 2)$ and concave upward on the interval $(2, \infty)$

d. Determine the x-value(s) of inflection point(s).

Using our results from part (c), we have that $y = x^3 - 6x^2 + 9x$ is a continuous function at $x = 2$, which changes concavity at $x = 2$ and so, has a point of inflection at $x = 2$, specifically at the point $(2, 2)$

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8. Use the following steps to graph $f(x) = \dfrac{x^2+1}{x^2-2}$:

1. Locate x and y intercepts.

<u>y-intercept (when $x = 0$)</u>

$$f(x) = \frac{x^2+1}{x^2-2}$$

$$f(0) = \frac{(0)^2+1}{(0)^2-2} = -\frac{1}{2}$$

<u>x-intercept (when $y = 0$)</u>

$$f(x) = \frac{x^2+1}{x^2-2}$$

$$\frac{x^2+1}{x^2-2} = 0$$

$$x^2 + 1 = 0$$

$$x^2 = -1$$

$$x = \pm\sqrt{-1}$$

(No real solutions)
No x-intercepts

2. Determine any horizontal or vertical asymptotes.

<u>Vertical asymptotes</u>

$f(x)$ is a rational function (meaning a fractional expression where the numerator and denominator are both polynomials). Rational functions have vertical asymptotes at x-values that cause the denominator to be zero (but not the numerator, also).

$$f(x)=\frac{x^2+1}{x^2-2}$$

$$x^2-2=0$$

$$x=\pm\sqrt{2}$$

Since we know the values that cause the numerator to be zero are imaginary, we need not worry about eliminating some of our solutions.

And so, $f(x)$ has vertical asymptotes at $x=\sqrt{2}$ and $x=-\sqrt{2}$

Additional note:

Notice that we could verify the vertical asymptote behavior near the values $x=\sqrt{2}$ and $x=-\sqrt{2}$ with a few quick one-sided limits:

Near $x=\sqrt{2}$

$$f(x)=\frac{x^2+1}{x^2-2}$$

$$\lim_{x\to\sqrt{2}^+}\frac{x^2+1}{x^2-2}\to\frac{3}{0^+}\to+\infty$$

$$\lim_{x\to\sqrt{2}^-}\frac{x^2+1}{x^2-2}\to\frac{3}{0^-}\to-\infty$$

Near $x=-\sqrt{2}$

$$f(x)=\frac{x^2+1}{x^2-2}$$

$$\lim_{x\to-\sqrt{2}^+}\frac{x^2+1}{x^2-2}\to\frac{3}{0^+}\to+\infty$$

$$\lim_{x\to-\sqrt{2}^-}\frac{x^2+1}{x^2-2}\to\frac{3}{0^-}\to-\infty$$

Horizontal asymptotes

Horizontal asymptotes occur for very large positive values of x or very large negative values of x. That is, far off to the right and far off to the left in our graph.

Therefore, they are determined using a limits at infinity:

As $x\to\infty$

$$f(x)=\frac{x^2+1}{x^2-2}$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{x^2+1}{x^2-2}$$

$$=\lim_{x\to\infty}\frac{x^2+1}{x^2-2}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)$$

$$=\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1-\frac{2}{x^2}}\to\frac{1+0}{1-0}=1$$

As $x\to-\infty$

$$f(x)=\frac{x^2+1}{x^2-2}$$

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{x^2+1}{x^2-2}$$

$$=\lim_{x\to-\infty}\frac{x^2+1}{x^2-2}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)$$

$$=\lim_{x\to-\infty}\frac{1+\frac{1}{x^2}}{1-\frac{2}{x^2}}\to\frac{1+0}{1-0}=1$$

And so, $f(x)$ has a HA of $y=1$ as $x\to\infty$ and as $x\to-\infty$

3. Find intervals where $f(x)$ is increasing and where it is decreasing.

Using the first derivative test, we have…

$$f(x)=\frac{x^2+1}{x^2-2}$$

$$f'(x)=\frac{\left(x^2-2\right)\left[x^2+1\right]'-\left(x^2+1\right)\left[x^2-2\right]'}{\left(x^2-2\right)^2}$$

$$f'(x)=\frac{\left(x^2-2\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-2\right)^2}$$

$$f'(x)=\frac{-6x}{\left(x^2-2\right)^2}$$

$\underline{f'(x)=0}$

$$\frac{-6x}{\left(x^2-2\right)^2}=0$$

$$-6x=0$$

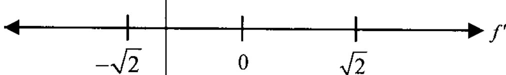
$$x=0$$

$\underline{f'(x)\text{ undefined}}$

$\dfrac{-6x}{\left(x^2-2\right)^2}$ is undefined where $\left(x^2-2\right)^2=0$

That is, $x=\sqrt{2}$ and $x=-\sqrt{2}$

Using a numberline to record the results…



Plugging in sample values…

$$f'(x)=\frac{-6x}{\left(x^2-2\right)^2}$$

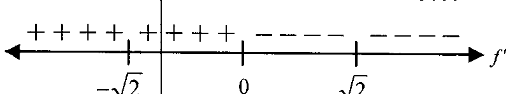
$$f'(-2)=\frac{-6(-2)}{\left((-2)^2-2\right)^2}=\frac{(+)}{(+)}=+$$

$$f'(-1)=\frac{-6(-1)}{\left((-1)^2-2\right)^2}=\frac{(+)}{(+)}=+$$

$$f'(1)=\frac{-6(1)}{\left((1)^2-2\right)^2}=\frac{(-)}{(+)}=-$$

$$f'(2)=\frac{-6(2)}{\left((2)^2-2\right)^2}=\frac{(-)}{(+)}=-$$

And so our numberline would look like…



Therefore, the function $f(x)=\frac{x^2+1}{x^2-2}$ is increasing on the interval

$\left(-\infty,-\sqrt{2}\right)\cup\left(-\sqrt{2},0\right)$

and decreasing on the interval $\left(0,\sqrt{2}\right)\cup\left(\sqrt{2},\infty\right)$

4. Locate all critical points and identify them as a max, a min, or neither.

Critical points are values in the domain of the function, which make the first derivative either equal zero, or undefined.

$x=0$ is in the domain of $f(x)=\frac{x^2+1}{x^2-2}$ and makes the first derivative equal zero.

So, we have a critical point at $x=0$.

Since the function increases to the left of $x=0$ and decreases to the right of $x=0$, we have a local maximum at $x=0$, where the graph reaches the point $\left(0,f(0)\right)=\left(0,-\frac{1}{2}\right)$

Additional: we have a local max at $x=0$, but not an absolute max, since the function went to positive infinity near some of the asymptotes, for example: $\lim_{x\to\sqrt{2}^+}\frac{x^2+1}{x^2-2}=+\infty$

Notice finally, that the values $x=\sqrt{2}$ and $x=-\sqrt{2}$ make the first derivative undefined, but are not in the domain of f, and so, are not critical points.

5. Find intervals where $f(x)$ is concave up and where it is concave down.

Using the second derivative test, we have...

$$f'(x)=\frac{-6x}{\left(x^2-2\right)^2}$$

$$f''(x)=\frac{\left(x^2-2\right)^2\left[-6x\right]'-\left(-6x\right)\left[\left(x^2-2\right)^2\right]'}{\left(\left(x^2-2\right)^2\right)^2}$$

$$f''(x)=\frac{\left(x^2-2\right)^2\left(-6\right)-\left(-6x\right)\left(2\left(x^2-2\right)\left(2x\right)\right)}{\left(x^2-2\right)^4}$$

$$f''(x)=\frac{6\left(x^2-2\right)\left\{-\left(x^2-2\right)+x\left(4x\right)\right\}}{\left(x^2-2\right)^4}$$

$$f''(x)=\frac{6\left(3x^2+2\right)}{\left(x^2-2\right)^3}$$

$$f''(x) = 0$$

$$\frac{6(3x^2+2)}{(x^2-2)^3} = 0$$

$$6(3x^2+2) = 0$$

$$x = \pm\sqrt{-\frac{2}{3}}$$

(No real solutions)
No values here.

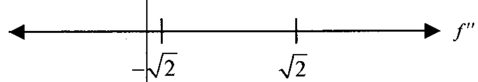
$$f''(x) \text{ undefined}$$

$$\frac{6(3x^2+2)}{(x^2-2)^3} \text{ is undefined where}$$

$$(x^2-2)^3 = 0$$

That is, $x = \sqrt{2}$ and $x = -\sqrt{2}$

Using a numberline to record the results…



Plugging in sample values…

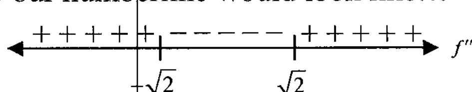
$$f''(x) = \frac{6(3x^2+2)}{(x^2-2)^3}$$

$$f''(-2) = \frac{6(3(-2)^2+2)}{((-2)^2-2)^3} = \frac{(+)}{(+)} = +$$

$$f''(0) = \frac{6(3(0)^2+2)}{((0)^2-2)^3} = \frac{(+)}{(-)} = -$$

$$f''(2) = \frac{6(3(2)^2+2)}{((2)^2-2)^3} = \frac{(+)}{(+)} = +$$

And so our numberline would look like…



Therefore, the function $f(x) = \frac{x^2+1}{x^2-2}$ is concave upward on the interval

$$(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$$

and concave downward on the interval $(-\sqrt{2}, \sqrt{2})$

6. Locate all inflection points.

Points of inflection are x-values in the domain of the function where the function changes concavity. While we have that f changes concavity at $x = \sqrt{2}$ and $x = -\sqrt{2}$, those values are not in the domain of $f(x) = \dfrac{x^2+1}{x^2-2}$

And so, we have no points of inflection for this function.

7. Graph the function below:

Putting together the results from parts (1) – (6), we have: