Calculus Refresher for M408D – Day 3 – Solutions

Evaluate the following integrals.

1.

$$\int \frac{x^2+2}{x^2}dx = \int\left(\frac{x^2}{x^2}+\frac{2}{x^2}\right)dx$$

$$= \int\left(1+2x^{-2}\right)dx$$

$$= x+\frac{2x^{-1}}{-1}+C$$

$$= x-\frac{2}{x}+C$$

2.

$$\int \sin^2 3x \cos 3x dx$$

$$= \int (\sin 3x)^2 (\cos 3x \; dx)$$

Let $u = \sin 3x$

$$= \int (u)^2 \left(\frac{du}{3}\right)$$

$du = \cos 3x \cdot 3 \; dx$

$$= \frac{1}{3}\int u^2 du$$

$\frac{du}{3} = \cos 3x \; dx$

$$= \frac{1}{3}\left(\frac{u^3}{3}\right)+C$$

$$= \frac{1}{9}\sin^3 3x + C$$

3.

$$\int \frac{1}{x\ln x}dx$$

$$= \int \frac{1}{x}\cdot\frac{1}{\ln x}\; dx$$

Let $u = \ln x$

$$= \int \frac{1}{u}\; du$$

$du = \frac{1}{x}\; dx$

$$= \ln|u| + C$$

$$= \ln|\ln x| + C$$

4.

$$\int \frac{3}{x^2+2x+1}\,dx$$

$$= 3\int \frac{1}{x^2+2x+1}\,dx$$

$$= 3\int \frac{1}{(x+1)^2}\,dx \qquad \text{Let } u = x+1$$

$$= 3\int \frac{1}{u^2}\,du \qquad du = dx$$

$$= 3\int u^{-2}\,du$$

$$= 3\left(\frac{u^{-1}}{-1}\right) + C$$

$$= -\frac{3}{x+1} + C$$

5.

$$\int_{-\pi}^{\pi} \sin^2 x\,dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{1}{2}(1-\cos 2x)\right)dx \qquad \text{Recall the identity: } \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2x)\,dx \qquad \text{Let } u = 2x \quad ; \quad u = -\pi \to x = -\frac{\pi}{2}$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos u)\left(\frac{du}{2}\right) \qquad du = 2dx \quad ; \quad u = \pi \to x = \frac{\pi}{2}$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos u)\,du$$

$$= \frac{1}{4}\left\{(u-\sin u)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right\}$$

$$= \frac{1}{4}\left\{\left(\frac{\pi}{2}-\sin\left(\frac{\pi}{2}\right)\right)-\left(-\frac{\pi}{2}-\sin\left(-\frac{\pi}{2}\right)\right)\right\}$$

$$= \frac{1}{4}\left\{\frac{\pi}{2}+\frac{\pi}{2}-\sin\frac{\pi}{2}+\sin\left(-\frac{\pi}{2}\right)\right\}$$

$$= \frac{1}{4}\{\pi-1+(-1)\}$$

$$= \frac{1}{4}\{\pi-2\}$$

6.

$$\int xe^{(x^2-1)}dx$$

$$=\int e^u\left(\frac{du}{2}\right) \qquad \text{Let } u = x^2-1$$

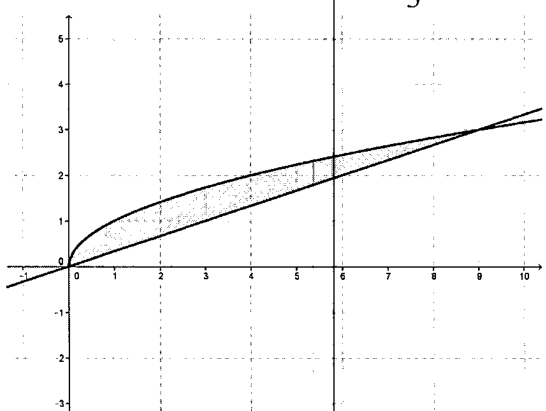
$$=\frac{1}{2}\int e^u du \qquad du = 2xdx$$

$$=\frac{1}{2}e^u + C \qquad \frac{du}{2} = xdx$$

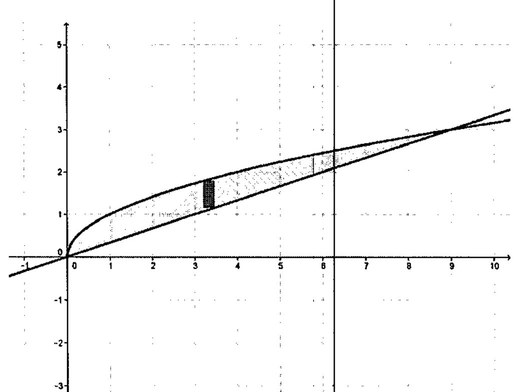
$$=\frac{1}{2}e^{x^2-1} + C$$

7. A) Write TWO definite integrals (one with respect to x, the other with respect to y) which represent the area, Ω, bound between the graphs of f and g

$f(x)=\sqrt{x} \qquad g(x)=\frac{1}{3}x$



With respect to x, we imagine the area as the sum of vertical rectangular strips.



The integral represents the sum of the areas of these rectangular strips.

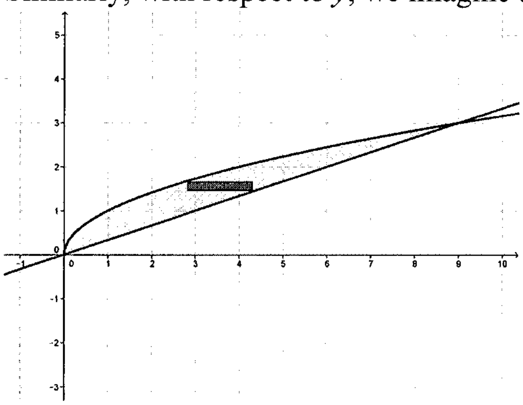
$\int_{um}$ (Areas of rectangles)

$\int_{um}$ (Heights) $\cdot$ (Widths)

$\int_{x=0}^{x=9} \left(f(x) - g(x)\right) \cdot dx$

$\int_0^9 \left(\sqrt{x} - \frac{1}{3}x\right) dx$

Similarly, with respect to y, we imagine the area to be composed of horizontal rectangular strips.



$\int_{um}$ (Areas of rectangles)

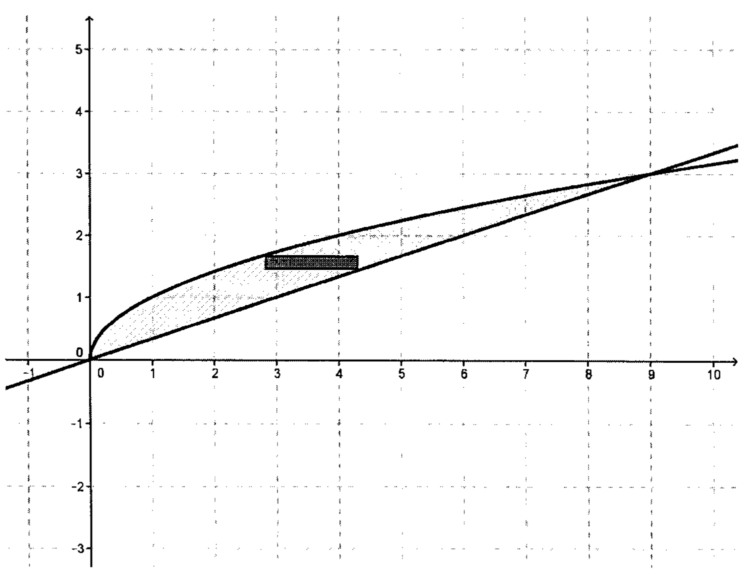
$\int_{um}$ (Lengths) $\cdot$ (Widths)

$\int_{y=0}^{y=3} \left(g(y) - f(y)\right) \cdot dy$

$g(x) = \frac{1}{3}x \quad ; \quad y = \frac{1}{3}x \quad ; \quad 3y = x \quad ; \quad g(y) = 3y$

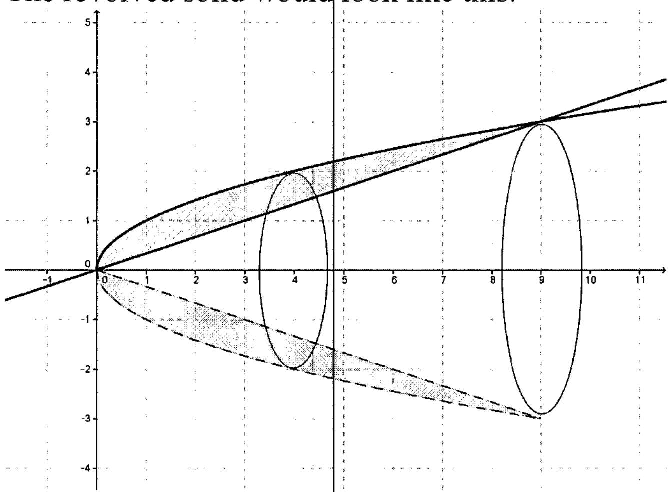
$f(x) = \sqrt{x} \quad ; \quad y = \sqrt{x} \quad ; \quad y^2 = x \ \ (\text{when } y > 0) \quad ; \quad f(y) = y^2 \ \ (\text{when } y > 0)$

$\int_0^3 \left(3y - y^2\right) dy$

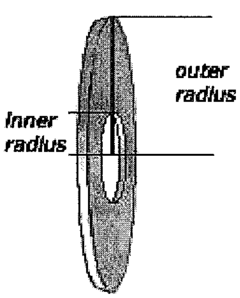


B) Find the volume of the solid generated by rotating Ω about the x-axis

The revolved solid would look like this:



Cross sections perpendicular to the x-axis would be "washers"



The volume of the solid can then be thought of as summing up the volume of each individual slice. If R = the outer radius and r = the inner radius, the volume of a washer of thickness h is given by:

$$V = \pi R^2 h - \pi r^2 h$$
$$= \pi\left(R^2 - r^2\right)h$$

The thickness h is a very small change in the x-direction, therefore dx

The outer radius R is the height of the function $f(x)$

The inner radius r is the height of the function $g(x)$

And so, we have:

$$\int_{um} (\text{Volumes of washers})$$

$$\int_{um} \pi\left(R^2 - r^2\right)h$$

$$\int_{x=0}^{x=9} \pi\left(\left(f(x)\right)^2 - \left(g(x)\right)^2\right)dx$$

$$\int_0^9 \pi\left(\left(\sqrt{x}\right)^2 - \left(\frac{1}{3}x\right)^2\right)dx$$

$$= \pi\int_0^9 \left(x - \frac{1}{9}x^2\right)dx$$

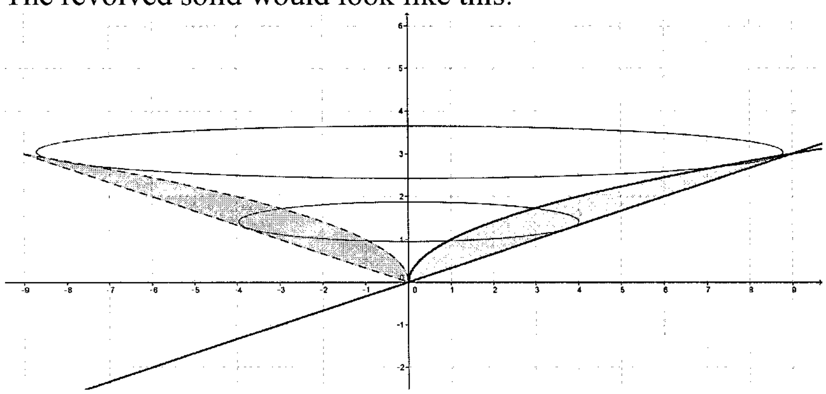
$$= \pi\left\{\left(\frac{1}{2}x^2 - \frac{1}{27}x^3\right)\Bigg|_0^9\right\}$$

$$= \pi\left\{\left(\frac{1}{2}(9)^2 - \frac{1}{27}(9)^3\right) - \left(\frac{1}{2}(0)^2 - \frac{1}{27}(0)^3\right)\right\}$$

$$= \frac{27\pi}{2}$$

C) Determine a definite integral which represents the volume of the solid generated by rotating the area Ω about the y-axis.

The revolved solid would look like this:



Similar to part (b), cross-sections perpendicular to the axis of revolution are "washers".
And the volume of any given washer is given by $V = \pi\left(R^2 - r^2\right)h$
The thickness h is a very small change in the y-direction, therefore dy
The outer radius R is the function $g(y)$
The inner radius r is the function $f(y)$
And so, we have:

$\int_{um}$ (Volumes of washers)

$\int_{um} \pi\left(R^2 - r^2\right)h$

$\int_{y=0}^{y=3} \pi\left(\left(g(y)\right)^2 - \left(f(y)\right)^2\right)dy$

$g(x) = \frac{1}{3}x \quad ; \quad y = \frac{1}{3}x \quad ; \quad 3y = x \quad ; \quad g(y) = 3y$

$f(x) = \sqrt{x} \quad ; \quad y = \sqrt{x} \quad ; \quad y^2 = x \quad (\text{when } y > 0) \quad ; \quad f(y) = y^2$

$\int_0^3 \pi\left((3y)^2 - \left(y^2\right)^2\right)dy$

$= \pi\int_0^3 \left(9y^2 - y^4\right)dy$

$= \pi\left\{\left(3y^3 - \frac{1}{5}y^5\right)\Big|_0^3\right\}$

$= \pi\left\{\left(3(3)^3 - \frac{1}{5}(3)^5\right) - \left(3(0)^3 - \frac{1}{5}(0)^5\right)\right\}$

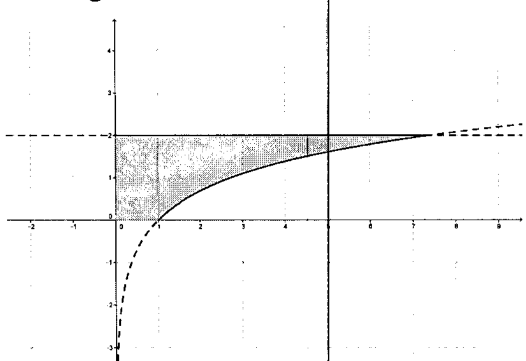
$= \frac{162\pi}{5}$

8. Find the volume obtained by rotating the region bounded by

$$y = \ln x, \quad y = 2, \quad x = 0, \quad y = 0$$

about the y-axis

The region to be rotated looks like:



And after rotation, the solid looks like:

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

$$\int e^x \sin x\, dx + \int e^x \sin x\, dx = -e^x \cos x + e^x \sin x + C$$

$$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x + C$$

$$\int e^x \sin x\, dx = \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right) + C$$

$$\int e^x \sin x\, dx = \frac{e^x}{2}\left(\sin x - \cos x\right) + C$$

Trigonometric Integrals practice:

13. $\int \sin^3 x \cos^2 x\, dx$

When you have powers of sine and cosine and if either of the powers is odd, generally, you can try to isolate one of terms that has the odd power:

$\int \sin^3 x \cos^2 x\, dx$

$= \int \sin x\left(\sin^2 x \cos^2 x\right) dx$

$= \int \sin x\left(\left(1-\cos^2 x\right)\cos^2 x\right) dx$ Using the identity $\sin^2 x + \cos^2 x = 1$

$= \int \sin x\left(\cos^2 x - \cos^4 x\right) dx$ Let $u = \cos x$

$= \int \left(u^2 - u^4\right)\left(-du\right)$ $du = -\sin x\, dx$

$= \int \left(u^4 - u^2\right) du$ $-du = \sin x\, dx$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

14. $\int \cos^2 x \sin 2x \, dx$

When the arguments of the trig functions don't match, we need to look to trigonometric identities to try to find a substitution to make them match.

$\int \cos^2 x \sin 2x \, dx$ Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$

$= \int \cos^2 x (2\sin x \cos x) \, dx$

$= \int 2\sin x \cos^3 x \, dx$

$= 2\int (u)^3 (-du)$ Let $u = \cos x$

$= -2\int u^3 du$ $du = -\sin x \, dx$

$= \dfrac{-2}{4} u^4 + C$ $-du = \sin x \, dx$

$= -\dfrac{1}{2}\cos^4 x + C$

15. $\int \tan^3 x \sec x \, dx$

The derivative of $\tan x$ is $\sec^2 x$ and the derivative of $\sec x$ is $\sec x \tan x$

So, a simple *u*-substitution doesn't appear to solve this one. After separating the terms in a different way, and recalling the trig identity for $\tan^2 x$, then the pieces start to come together…

$\int \tan^3 x \sec x \, dx$

$= \int \tan x (\tan^2 x) \sec x \, dx$ Using the identity $\tan^2\theta = \sec^2\theta - 1$

$= \int \tan x (\sec^2 x - 1) \sec x \, dx$

$= \int (\sec^2 x - 1) \sec x \tan x \, dx$ Let $u = \sec x$

$= \int (u^2 - 1)(du)$ $du = \sec x \tan x \, dx$

$= \dfrac{1}{3}u^3 - u + C$

$= \dfrac{1}{3}\sec^3 x - \sec x + C$