Calculus Refresher for M408D – Day 1 – Solutions

1. For the function defined by:

$$f(x)=\begin{cases}\dfrac{3}{x+5} & x<-1\\ 3\left(\dfrac{x+2}{3-x}\right) & -1<x<0\\ \dfrac{x^2-2x+1}{x-1} & x>0\end{cases}$$

Determine the following limits:

A) $\lim_{x\to -1} f(x)$

B) $\lim_{x\to 0} f(x)$

A) We recall that a limit exists at a value $x=a$ if and only if the limit from the left exists, the limit from the right exists, and those two limits match.

That is, $\lim_{x\to a} f(x)=L \quad \Leftrightarrow \quad \begin{cases}\lim_{x\to a^+} f(x)=L\\ \text{and}\\ \lim_{x\to a^-} f(x)=L\end{cases}$

In the given piece-wise function, we have...

$$\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}\left(\frac{3}{x+5}\right)=\left(\frac{3}{(-1)+5}\right)=\frac{3}{4}$$

(Since the expression $\dfrac{3}{x+5}$ is continuous near $x=-1$, we can evaluate the limit by "plugging in".)

And $\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+} 3\left(\dfrac{x+2}{3-x}\right)=3\left(\dfrac{(-1)+2}{3-(-1)}\right)=\dfrac{3}{4}$

So $\lim_{x\to -1} f(x)=\dfrac{3}{4}$

B) Similar to part A, we check the limits from both sides...

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} 3\left(\frac{x+2}{3-x}\right)=3\left(\frac{(0)+2}{3-(0)}\right)=2$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x^2-2x+1}{x-1}=\frac{1}{-1}=-1$$

Since these don't match, we have that $\lim_{x\to 0} f(x)$ does not exist

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2. Find the limit if it exists.

A) Testing the limit by plugging in, we have that $\lim_{x\to 7}\dfrac{\sqrt{x+2}\ -3}{x-7}\to\dfrac{0}{0}$ which is an indeterminate form that indicates we have more work to do.

When we were first learning limits, we would be forced to use algebraic techniques such as multiplying by a form of one using the conjugate of the numerator:

$$\lim_{x\to 7}\frac{\sqrt{x+2}\ -3}{x-7}\cdot\left(\frac{\sqrt{x+2}\ +3}{\sqrt{x+2}\ +3}\right)=\lim_{x\to 7}\frac{(x+2)-9}{(x-7)\left(\sqrt{x+2}\ +3\right)}$$
$$=\lim_{x\to 7}\frac{x-7}{(x-7)\left(\sqrt{x+2}\ +3\right)}$$
$$=\lim_{x\to 7}\frac{1}{\left(\sqrt{x+2}\ +3\right)}=\frac{1}{6}$$

But now that we know L'Hospital's rule, and that $\frac{0}{0}$ is one of the two L'Hospital forms (the other being $\frac{\pm\infty}{\pm\infty}$), we could have used it as well:

$$\lim_{x\to 7}\frac{\sqrt{x+2}\ -3}{x-7}\ \underset{=\!=}{\overset{L'H}{}}\ \lim_{x\to 7}\frac{\frac{d}{dx}\left[\sqrt{x+2}\ -3\right]}{\frac{d}{dx}\left[x-7\right]}$$
$$=\lim_{x\to 7}\frac{\frac{1}{2}(x+2)^{-1/2}}{1}$$
$$=\lim_{x\to 7}\frac{1}{2\sqrt{x+2}}=\frac{1}{6}$$

B) $\lim_{t\to 0}\left(\frac{1}{t}\ -\ \frac{1}{t^2+t}\right)$

Start by taking a look at this problem from one side of zero, say the positive side.

Testing the behavior of the limit by plugging in, we have $\lim_{t\to 0^+}\left(\frac{1}{t}\ -\ \frac{1}{t^2+t}\right)\to\infty-\infty$ which is an indeterminate form.

Using a little algebra to combine the terms, we have

$$\lim_{t\to 0^+}\left(\frac{1}{t}\ -\ \frac{1}{t^2+t}\right)=\lim_{t\to 0^+}\left(\frac{t+1-1}{t(t+1)}\right)$$
$$=\lim_{t\to 0^+}\left(\frac{t}{t(t+1)}\right)$$
$$=\lim_{t\to 0^+}\left(\frac{1}{t+1}\right)=1$$

The process is similar to show that the limit from the negative side is also one.

$$\lim_{t\to 0^-}\left(\frac{1}{t}\ -\ \frac{1}{t^2+t}\right)=1$$

Therefore, $\lim_{t\to 0}\left(\frac{1}{t}\ -\ \frac{1}{t^2+t}\right)=1$

C) Testing the limit by plugging in, we have $\lim\limits_{x\to-1}\frac{x^2-4x}{x^2-3x-4}\to\frac{5}{0}$

This result tells us that the limit may be $+\infty$ (if the expression goes to infinity as you approach -1 from both sides), $-\infty$ (again if this is the behavior from both sides), or "Does Not Exist" (most commonly, if the expression goes to positive infinity from one side and negative infinity from the other).

Using a little algebra, we have

$$\frac{x^2-4x}{x^2-3x-4}=\frac{x(x-4)}{(x+1)(x-4)}=\frac{x}{x+1}\quad(\text{when } x\neq 4)$$

Since we are looking at the function's behavior near -1 and we're not at $x=4$, we can write:

$$\lim_{x\to-1}\frac{x^2-4x}{x^2-3x-4}=\lim_{x\to-1}\frac{x}{x+1}$$

And so, testing from the positive side of -1...

$$\lim_{x\to-1^+}\frac{x^2-4x}{x^2-3x-4}=\lim_{x\to-1^+}\frac{x}{x+1}\to\frac{-1}{0^+}\to-\infty$$

And from the negative side...

$$\lim_{x\to-1^-}\frac{x^2-4x}{x^2-3x-4}=\lim_{x\to-1^-}\frac{x}{x+1}\to\frac{-1}{0^-}\to+\infty$$

And so, $\lim\limits_{x\to-1}\frac{x^2-4x}{x^2-3x-4}$ Does Not Exist (DNE)

D) $\lim\limits_{x\to2^-}\frac{|x-2|}{x-2}$

Taking a look at the numerator, it will be advantageous to look at the absolute value function as a piece-wise function.

Recall that all absolute value expressions can be derived starting with the basic one:

$$|A|=\begin{cases}A & \text{when } A\geq 0\\ -A & \text{when } A<0\end{cases}$$

$$|x-2|=\begin{cases}x-2 & x-2\geq 0\\ -(x-2) & x-2<0\end{cases}\quad\text{or}\quad |x-2|=\begin{cases}x-2 & x\geq 2\\ -(x-2) & x<2\end{cases}$$

And so, since our limit is approaching 2 from the negative side, we can write

$$\lim_{x\to2^-}\frac{|x-2|}{x-2}=\lim_{x\to2^-}\frac{-(x-2)}{x-2}=\lim_{x\to2^-}(-1)=-1$$

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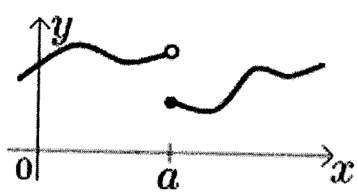
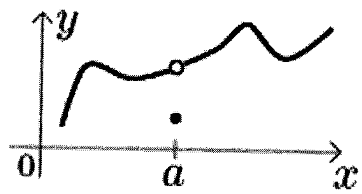
3. A) State the three types of discontinuities and sketch a picture of each.

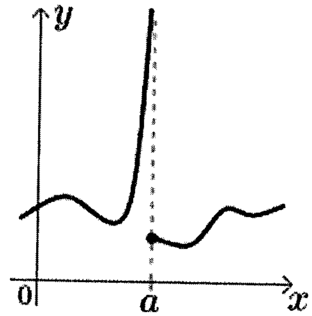
Jump Discontinuity



Removable Discontinuity



Infinite Discontinuity



B) Which discontinuities, if any, exist for the piece-wise function f in problem 1?

In the interval $(-\infty,-1)$, the function is defined by $\dfrac{3}{x+5}$ which has a vertical asymptote at $x=-5$ and therefore an infinite discontinuity there.

At $x=-1$ we have that $\lim\limits_{x\to-1^-} f(x)=\lim\limits_{x\to-1^+} f(x)=\dfrac{3}{4}$

However, f is not defined at $x=-1$ and so we have a "hole", that is, a removable discontinuity.

In the interval $(-1,0)$, the function is defined by $3\left(\dfrac{x+2}{3-x}\right)$ which is continuous everywhere except $x=3$, which is not in the interval $(-1,0)$, so f is continuous on $(-1,0)$

At $x=0$ we have that $\lim\limits_{x\to0^-} f(x)=2$ and $\lim\limits_{x\to0^+} f(x)=-1$, therefore a jump discontinuity.

In the interval $(0,\infty)$, the function is defined by $\dfrac{x^2-2x+1}{x-1}$ which is continuous everywhere except $x=1$. Notice from the factorization of the numerator, that

$$\frac{x^2-2x+1}{x-1}=\frac{(x-1)(x-1)}{(x-1)}=x-1 \quad (\text{whenever } x\neq1)$$

Therefore, we have a "hole" at $x=1$, that is, a removable discontinuity.

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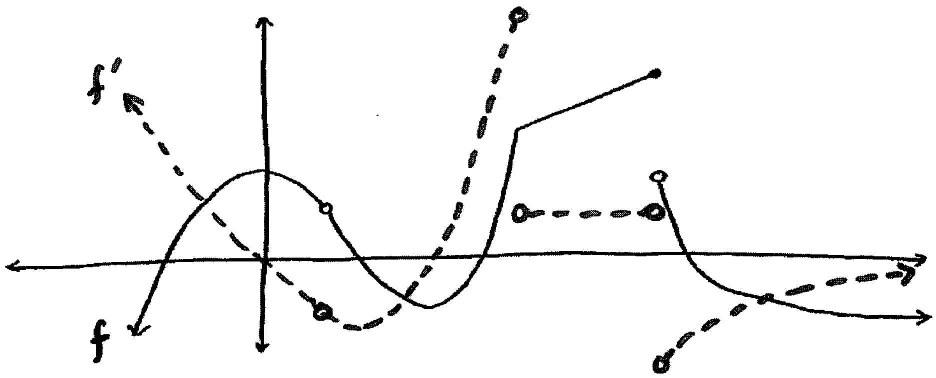
4. Draw the graph of f', given the following graph of f



Just a word of advice: I recommend thinking about the slope of the tangent line to f at various points and translating that to heights for f'
For instance, when wanting to draw the derivative you could start by putting a point at the origin because f appears to have a horizontal tangent line at $x = 0$
That is, "slope of tangent to f at zero" corresponds to "height of f' at zero"
Similarly, we can place a point on the x-axis where f has the local minimum.
After getting these reference points, we could say to ourselves, "okay, to the left of the origin, what values do the slopes of tangent lines to f have?"
They are positive, and as I move toward $x = 0$ from the left, the slopes of tangents approach zero.
Therefore the height of f' should be positive in this interval, then approach zero as I move toward $x = 0$ from the left.
And so on...

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5. Let

$$f(x) = \begin{Bmatrix} 2x^2 - 1 & x \le 1 \\ 4x - 3 & x > 1 \end{Bmatrix}$$

A) Is f continuous at $x = 1$?

B) Is f differentiable at $x = 1$?

A) Continuity, in a nutshell, is asking whether the following equality holds:
$\lim_{x\to 1} f(x) = f(1)$
To determine the value of the limit, we should investigate the limit as x approaches one from both sides...
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(2x^2 - 1) = 2(1)^2 - 1 = 1$
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(4x - 3) = 4(1) - 3 = 1$
And so, $\lim_{x\to 1} f(x) = 1$
Finally, the piecewise definition gives us that $f(1) = 2(1)^2 - 1 = 1$
And so, we see that $\lim_{x\to 1} f(x) = f(1)$
Therefore, f is continuous at $x = 1$

B) Continuity is necessary for a function to be differentiable, but it is not sufficient. We need to also check that the function does not have a "corner" or abrupt change in slope at $x=1$. Recall that a continuous function can have places such as a corner, cusp or vertical tangent line, where a derivative does not exist.

A corner or vertical tangent can be detected by looking at what values the slope of the tangent line approaches as we approach $x=1$ from either side.

To do this, we use the one-sided derivatives:

$$f'_-(a)=\lim_{h\to 0^-}\frac{f(a+h)-f(a)}{h} \quad \text{and} \quad f'_+(a)=\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$$

But we need not compute them by way of the difference quotients.

Instead, we can say that for $x=1$

$$f'_-(x)=\frac{d}{dx}\left[2x^2-1\right]=4x$$

And so $f'_-(1)=4(1)=4$

Similarly, $f'_+(x)=\frac{d}{dx}[4x-3]=4$

And so $f'_+(1)=4$

Therefore, we have that the slopes match on each side of $x=1$, and so the graph comes together with no "corner".

And so, f is differentiable at $x=1$

----------------Additional info-------------

We could have also used the other definition of derivative and those one-sided derivative definitions.

$$f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

We would want to compute and compare the two values:

$$f'_-(a)=\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a} \quad \text{and} \quad f'_+(a)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

$$\begin{aligned}\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}&=\lim_{x\to 1^-}\frac{\left(2(x)^2-1\right)-\left(2(1)^2-1\right)}{x-1}\\&=\lim_{x\to 1^-}\frac{2x^2-2}{x-1}\\&=\lim_{x\to 1^+}\frac{2(x-1)(x+1)}{x-1}\\&=\lim_{x\to 1^-}\left(2(x+1)\right)=4\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}&=\lim_{x\to 1^+}\frac{\left(4(x)-3\right)-\left(2(1)^2-1\right)}{x-1}\\&=\lim_{x\to 1^+}\frac{4(x-1)}{x-1}\\&=\lim_{x\to 1^-}4=4\end{aligned}$$

Since these two on-sided derivatives match, the derivative exists at $x=1$ and equals 4

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6. Find all x-values where the tangent line to $f(x)=\frac{1}{3}x^3-x^2-13x+7\pi$ has a slope of 2.

The derivative gives the slope of the tangent line to the graph of $f(x)$

Therefore, we're looking for the x-values where $f'(x)=2$

$$f'(x)=x^2-2x-13$$
$$2=x^2-2x-13$$
$$0=x^2-2x-15$$
$$0=(x-5)(x+3)$$
$$x=5\ ;\ x=-3$$

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7. Find the first derivative (dy/dx):

A)

$$y=\frac{3}{(2x^2+5x)^{3/2}}=3(2x^2+5x)^{-3/2}$$
$$\frac{dy}{dx}=-\frac{9}{2}(2x^2+5x)^{-5/2}\left[2x^2+5x\right]'$$
$$\frac{dy}{dx}=\frac{-9(4x+5)}{2(2x^2+5x)^{5/2}}$$

B)

$$y=\sec^2(5x)=(\sec 5x)^2$$
$$\frac{dy}{dx}=2(\sec 5x)^1\left[\sec 5x\right]'$$
$$\frac{dy}{dx}=2(\sec 5x)^1(\sec 5x\tan 5x(5))$$
$$\frac{dy}{dx}=10\sec^2 5x\tan 5x$$

C)

$$y=\sqrt{\frac{1-x}{1+x^2}}=\left(\frac{1-x}{1+x^2}\right)^{-1/2}$$

$$\frac{dy}{dx}=-\frac{1}{2}\left(\frac{1-x}{1+x^2}\right)^{-3/2}\left[\frac{1-x}{1+x^2}\right]'$$

$$\frac{dy}{dx}=-\frac{1}{2}\left(\frac{1-x}{1+x^2}\right)^{-3/2}\frac{(1+x^2)(-1)-(1-x)(2x)}{(1+x^2)^2}$$

$$\frac{dy}{dx}=-\frac{1}{2}\left(\frac{1-x}{1+x^2}\right)^{-3/2}\frac{x^2-2x-1}{(1+x^2)^2}$$

$$\frac{dy}{dx}=-\frac{1}{2}\cdot\frac{(1+x^2)^{3/2}}{(1-x)^{3/2}}\cdot\frac{x^2-2x-1}{(1+x^2)^2}$$

$$\frac{dy}{dx}=-\frac{x^2-2x-1}{2(1-x)^{3/2}(1+x^2)^{1/2}}$$

D)

$$y=\ln\left(3x^2+4x\right)$$

$$\frac{dy}{dx}=\frac{1}{3x^2+4x}\left[3x^2+4x\right]'$$

$$\frac{dy}{dx}=\frac{6x+4}{3x^2+4x}$$

E)

$$y=2x-\frac{1}{2}e^{2x}$$

$$\frac{dy}{dx}=2-\frac{1}{2}e^{2x}\left[2x\right]'$$

$$\frac{dy}{dx}=2-e^{2x}$$

F)

$$\sin\left(y^2\right)=y\cos(x)$$

$$\cos\left(y^2\right)\frac{d}{dx}\left[y^2\right]=\frac{d}{dx}[y]\cdot(\cos x)+(y)\cdot\frac{d}{dx}[\cos x]$$

$$\cos\left(y^2\right)\left(2y\frac{dy}{dx}\right)=\frac{dy}{dx}\cos x+y(-\sin x)$$

$$\cos\left(y^2\right)\left(2y\frac{dy}{dx}\right)-\frac{dy}{dx}\cos x=-y\sin x$$

$$\left(2y\cos\left(y^2\right)-\cos x\right)\frac{dy}{dx}=-y\sin x$$

$$\frac{dy}{dx}=\frac{-y\sin x}{2y\cos\left(y^2\right)-\cos x}$$

$$\frac{dy}{dx}=\frac{y\sin x}{\cos x-2y\cos\left(y^2\right)}$$

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8. Find equation of tangent line to curve at the point $(1,2)$

$$x^2+2xy-y^2+x=2$$

The equation of a line can be determined by finding (1) a point on the line, (2) the slope of the line and then combining those ingredients in the point-slope formula $y-y_1=m(x-x_1)$

(1) A point on the line is given in the problem statement: $(1,2)$

(2) The slope of the line is given by the derivative evaluated at the point $(1,2)$

$$x^2+2xy-y^2+x=2$$

$$2x+2\left\{(1)y+(x)\frac{dy}{dx}\right\}-2y\frac{dy}{dx}+1=0$$

$$2x\frac{dy}{dx}-2y\frac{dy}{dx}=-2x-2y-1$$

$$\frac{dy}{dx}=\frac{2x+2y+1}{2y-2x}$$

$$\left.\frac{dy}{dx}\right|_{(1,2)}=\frac{2(1)+2(2)+1}{2(2)-2(1)}=\frac{7}{2}$$

And combining this information using the point-slope formula...

$$y-2=\frac{7}{2}(x-1)$$

$$y=\frac{7}{2}x-\frac{3}{2}$$

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9. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 24 cm.

The surface area of a sphere is given by: $S = 4\pi r^2$
In the problem statement we are given the rate of change of the surface area.
In symbols, we might then write: $\frac{dS}{dt} = -1 \text{ cm}^2/\text{min}$ (negative because the area is decreasing)
We are asked to find the rate of change of diameter. If we let $D = $ diameter, then in symbols, we are looking for $\frac{dD}{dt}$ when $D = 24$
Looking back to the original equation, $S = 4\pi r^2$, we see that differentiating it with respect to time, t, would give us the expression $\frac{dS}{dt}$ which is good, but would not produce the expression $\frac{dD}{dt}$ which we want. Therefore, we want to translate the surface area formula into a formula in terms of diameter:
Recall that $D = 2r$ and so, $r = \frac{D}{2}$

$$S = 4\pi r^2$$

$$S = 4\pi\left(\frac{D}{2}\right)^2$$

$$S = \pi D^2$$

Differentiating with respect to time, we have

$$\frac{dS}{dt} = 2\pi D \cdot \frac{dD}{dt}$$

And substituting in known values

$$\frac{dS}{dt} = 2\pi D \cdot \frac{dD}{dt}$$

$$(-1) = 2\pi(24) \cdot \frac{dD}{dt}$$

$$\frac{dD}{dt} = \frac{-1}{48\pi} \text{ cm/min}$$

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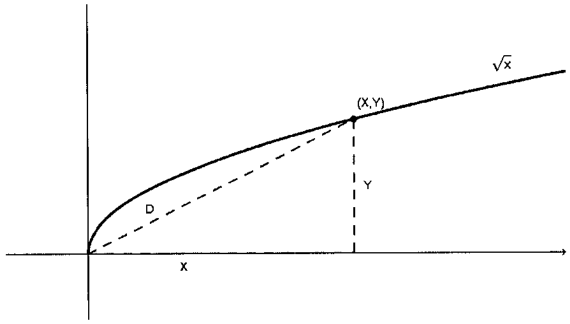
10. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point $(4, 2)$, its x-coordinate increases at a rate of 3 cm/sec. How fast is the distance from the particle to the origin changing at this instant?

If we use a graph to help organize the information given, we would have something like this:



If we let D represent the distance of a point (x,y) on the curve from the origin, the problem can be summarized as:

Determine $\frac{dD}{dt}$ when $\frac{dx}{dt}=3$ cm/sec and $x=4$; $y=2$

The distance of a point (x,y) from the origin is given by:

$$D=\sqrt{(x-0)^2+(y-0)^2}$$

$$D=\sqrt{x^2+y^2}$$

However, we see that if we differentiate this expression with respect to time, we will generate the term $\frac{dy}{dt}$ which is not given.

One option is to proceed by rewriting the distance function entirely in terms of x
We'll try that...

Notice that any point (x,y) on the curve $y=\sqrt{x}$ follows this equality, and so, those sets of points can be expressed as $\left(x,\sqrt{x}\right)$. In other words, the equation $y=\sqrt{x}$ can be used to substitute into the distance equation:

$$D=\sqrt{x^2+y^2}$$

$$D=\sqrt{x^2+\left(\sqrt{x}\right)^2}$$

$$D=\sqrt{x^2+x}$$

And now differentiating with respect to time, we have:

$$D=\sqrt{x^2+x}=\left(x^2+x\right)^{1/2}$$

$$\frac{dD}{dt}=\frac{1}{2}\left(x^2+x\right)^{-1/2}\left(2x\frac{dx}{dt}+\frac{dx}{dt}\right)$$

$$\frac{dD}{dt}=\frac{2x\frac{dx}{dt}+\frac{dx}{dt}}{2\sqrt{x^2+x}}$$

And so, evaluating $\frac{dD}{dt}$ when $\frac{dx}{dt}=3$ cm/sec and $x=4$, we have:

$$\frac{dD}{dt}=\frac{2x\frac{dx}{dt}+\frac{dx}{dt}}{2\sqrt{x^2+x}}$$

$$\frac{dD}{dt}=\frac{2(4)(3)+(3)}{2\sqrt{(4)^2+(4)}}=\frac{27}{4\sqrt{5}}\text{ cm/sec}$$

--------------------------Alternate solution----------------

Suppose back in our first stages with the equation $D=\sqrt{x^2+y^2}$ that we had decided to differentiate here.

$$D=\sqrt{x^2+y^2}=\left(x^2+y^2\right)^{1/2}$$

$$\frac{dD}{dt}=\frac{1}{2}\left(x^2+y^2\right)^{-1/2}\left(2x\frac{dx}{dt}+2y\frac{dy}{dt}\right)$$

$$\frac{dD}{dt}=\frac{2x\frac{dx}{dt}+2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}}$$

We originally wanted to determine the value of $\frac{dD}{dt}$ when $\frac{dx}{dt}=3$ cm/sec and $x=4$; $y=2$

However, using the expression we've derived above, we would also need to know the value of $\frac{dy}{dt}$ at the instance described.

Notice that we can get $\frac{dy}{dt}$ by differentiating the expression for the curve $y=\sqrt{x}$

$$y=\sqrt{x}$$

$$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}\frac{dx}{dt}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}\frac{dx}{dt}$$

And evaluating this expression when $\frac{dx}{dt}=3$ cm/sec and $x=4$ we get

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x}}\frac{dx}{dt}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{(4)}}(3)=\frac{3}{4}\text{ cm/sec}$$

And now substituting into our original derivation, we have

$$\frac{dD}{dt}=\frac{2x\frac{dx}{dt}+2y\frac{dy}{dt}}{2\sqrt{x^2+y^2}}$$

$$\frac{dD}{dt}=\frac{2(4)(3)+2(2)\left(\frac{3}{4}\right)}{2\sqrt{(4)^2+(2)^2}}=\frac{27}{4\sqrt{5}}\text{ cm/sec}$$