Calculus Refresher for M408D – Day 3 – Solutions

Evaluate the following integrals.

1.

$$\int \frac{x^2+2}{x^2}dx = \int\left(\frac{x^2}{x^2}+\frac{2}{x^2}\right)dx$$
$$= \int\left(1+2x^{-2}\right)dx$$
$$= x+\frac{2x^{-1}}{-1}+C$$
$$= x-\frac{2}{x}+C$$

2.

$$\int \sin^2 3x \cos 3x dx$$
$$= \int (\sin 3x)^2 (\cos 3x\ dx) \qquad \text{Let } u = \sin 3x$$
$$= \int (u)^2 \left(\frac{du}{3}\right) \qquad du = \cos 3x \cdot 3\ dx$$
$$= \frac{1}{3}\int u^2 du \qquad \frac{du}{3} = \cos 3x\ dx$$
$$= \frac{1}{3}\left(\frac{u^3}{3}\right)+C$$
$$= \frac{1}{9}\sin^3 3x + C$$

3.

$$\int \frac{1}{x\ln x}dx$$
$$= \int \frac{1}{x}\cdot\frac{1}{\ln x}\ dx \qquad \text{Let } u = \ln x$$
$$= \int \frac{1}{u}\ du \qquad du = \frac{1}{x}\ dx$$
$$= \ln|u| + C$$
$$= \ln|\ln x| + C$$

4.

$$\int \frac{3}{x^2+2x+1}\,dx$$

$$=3\int \frac{1}{x^2+2x+1}\,dx$$

$$=3\int \frac{1}{(x+1)^2}\,dx \qquad \text{Let } u = x+1$$

$$=3\int \frac{1}{u^2}\,du \qquad du = dx$$

$$=3\int u^{-2}\,du$$

$$=3\left(\frac{u^{-1}}{-1}\right)+C$$

$$=-\frac{3}{x+1}+C$$

5.

$$\int_{-\pi}^{\pi} \sin^2 x\,dx$$

$$=\int_{-\pi}^{\pi}\left(\frac{1}{2}(1-\cos 2x)\right)dx \qquad \text{Recall the identity: } \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2x)\,dx \qquad \text{Let } u = 2x \quad ; \quad u=-\pi \to x = -\frac{\pi}{2}$$

$$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos u)\left(\frac{du}{2}\right) \qquad du = 2dx \quad ; \quad u=\pi \to x=\frac{\pi}{2}$$

$$=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos u)\,du$$

$$=\frac{1}{4}\left\{(u-\sin u)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right\}$$

$$=\frac{1}{4}\left\{\left(\frac{\pi}{2}-\sin\left(\frac{\pi}{2}\right)\right)-\left(-\frac{\pi}{2}-\sin\left(-\frac{\pi}{2}\right)\right)\right\}$$

$$=\frac{1}{4}\left\{\frac{\pi}{2}+\frac{\pi}{2}-\sin\frac{\pi}{2}+\sin\left(-\frac{\pi}{2}\right)\right\}$$

$$=\frac{1}{4}\{\pi-1+(-1)\}$$

$$=\frac{1}{4}\{\pi-2\}$$

6.

$$\int xe^{\left(x^2-1\right)}dx$$

$$=\int e^u\left(\frac{du}{2}\right) \qquad \text{Let } u=x^2-1$$

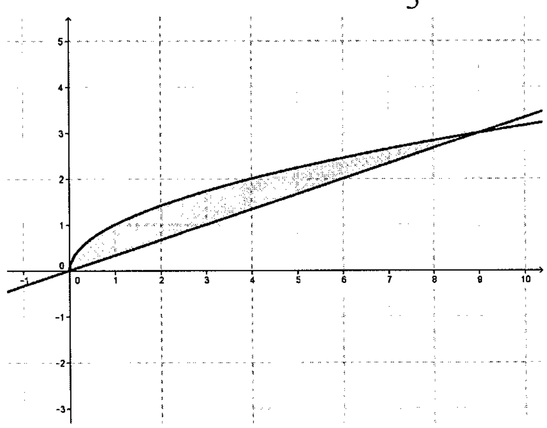
$$=\frac{1}{2}\int e^u du \qquad du=2xdx$$

$$=\frac{1}{2}e^u+C \qquad \frac{du}{2}=xdx$$

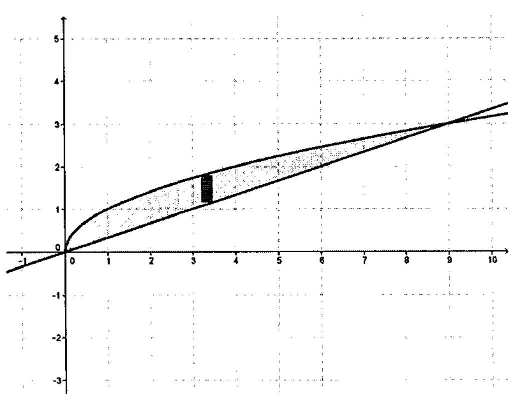
$$=\frac{1}{2}e^{x^2-1}+C$$

7. A) Write TWO definite integrals (one with respect to x, the other with respect to y) which represent the area, Ω, bound between the graphs of f and g

$f(x)=\sqrt{x} \qquad g(x)=\frac{1}{3}x$



With respect to x, we imagine the area as the sum of vertical rectangular strips.



The integral represents the sum of the areas of these rectangular strips.

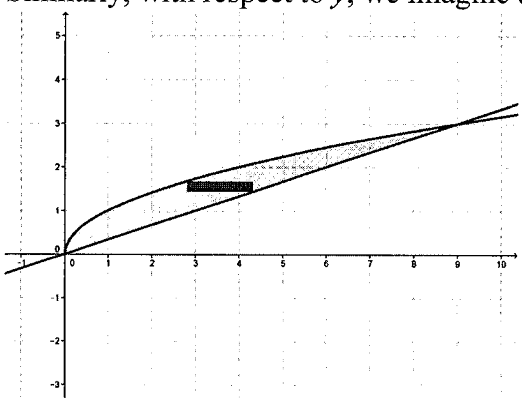
$\int_{um}(\text{Areas of rectangles})$

$\int_{um}(\text{Heights})\cdot(\text{Widths})$

$\int_{x=0}^{x=9}\left(f(x)-g(x)\right)\cdot dx$

$\int_0^9\left(\sqrt{x}-\frac{1}{3}x\right)dx$

Similarly, with respect to *y*, we imagine the area to be composed of horizontal rectangular strips.



$\int_{um}(\text{Areas of rectangles})$

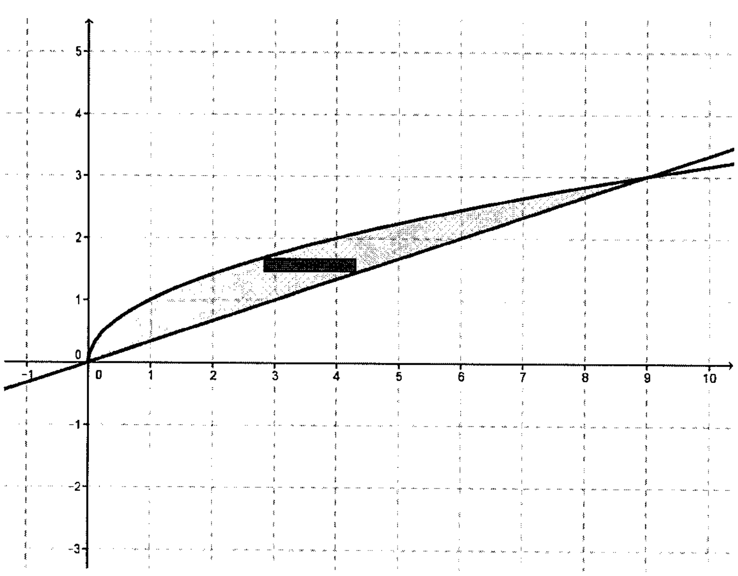
$\int_{um}(\text{Lengths})\cdot(\text{Widths})$

$\int_{y=0}^{y=3}\left(g(y)-f(y)\right)\cdot dy$

$g(x)=\frac{1}{3}x \quad ; \quad y=\frac{1}{3}x \quad ; \quad 3y=x \quad ; \quad g(y)=3y$

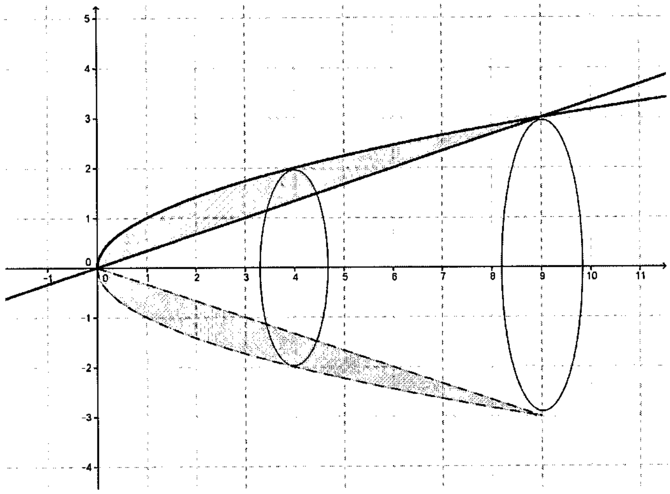
$f(x)=\sqrt{x} \quad ; \quad y=\sqrt{x} \quad ; \quad y^2=x \;\; (\text{when } y>0) \quad ; \quad f(y)=y^2 \;\; (\text{when } y>0)$

$\int_0^3\left(3y-y^2\right)dy$

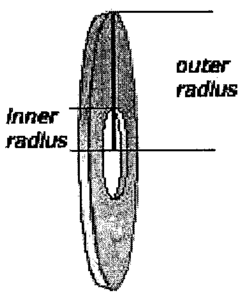


B) Find the volume of the solid generated by rotating Ω about the x-axis

The revolved solid would look like this:



Cross sections perpendicular to the x-axis would be "washers"



The volume of the solid can then be thought of as summing up the volume of each individual slice. If R = the outer radius and r = the inner radius, the volume of a washer of thickness h is given by:

$$V = \pi R^2 h - \pi r^2 h$$
$$= \pi\left(R^2 - r^2\right)h$$

The thickness h is a very small change in the x-direction, therefore dx

The outer radius R is the height of the function $f(x)$

The inner radius r is the height of the function $g(x)$

And so, we have:

$$\int_{um}(\text{Volumes of washers})$$

$$\int_{um}\pi\left(R^2-r^2\right)h$$

$$\int_{x=0}^{x=9}\pi\left((f(x))^2-(g(x))^2\right)dx$$

$$\int_0^9\pi\left(\left(\sqrt{x}\right)^2-\left(\frac{1}{3}x\right)^2\right)dx$$

$$=\pi\int_0^9\left(x-\frac{1}{9}x^2\right)dx$$

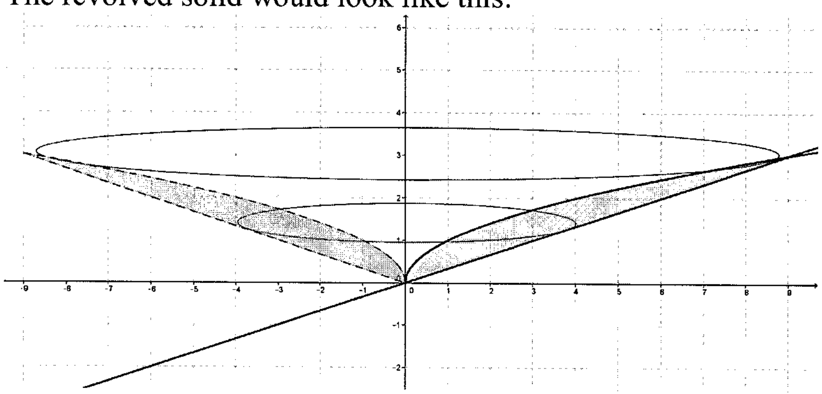
$$=\pi\left\{\left(\frac{1}{2}x^2-\frac{1}{27}x^3\right)\Bigg|_0^9\right\}$$

$$=\pi\left\{\left(\frac{1}{2}(9)^2-\frac{1}{27}(9)^3\right)-\left(\frac{1}{2}(0)^2-\frac{1}{27}(0)^3\right)\right\}$$

$$=\frac{27\pi}{2}$$

C) Determine a definite integral which represents the volume of the solid generated by rotating the area Ω about the y-axis.

The revolved solid would look like this:



Similar to part (b), cross-sections perpendicular to the axis of revolution are "washers".
And the volume of any given washer is given by $V=\pi\left(R^2-r^2\right)h$

The thickness h is a very small change in the y-direction, therefore dy

The outer radius R is the function $g(y)$

The inner radius r is the function $f(y)$

And so, we have:

$\int_{um} (\text{Volumes of washers})$

$\int_{um} \pi \left(R^2 - r^2\right) h$

$\int_{y=0}^{y=3} \pi \left(\left(g(y)\right)^2 - \left(f(y)\right)^2 \right) dy$

$g(x) = \frac{1}{3}x \quad ; \quad y = \frac{1}{3}x \quad ; \quad 3y = x \quad ; \quad g(y) = 3y$

$f(x) = \sqrt{x} \quad ; \quad y = \sqrt{x} \quad ; \quad y^2 = x \ \ (\text{when } y > 0) \quad ; \quad f(y) = y^2$

$\int_0^3 \pi \left((3y)^2 - \left(y^2\right)^2 \right) dy$

$= \pi \int_0^3 \left(9y^2 - y^4\right) dy$

$= \pi \left\{ \left(3y^3 - \frac{1}{5}y^5 \right) \Big|_0^3 \right\}$

$= \pi \left\{ \left(3(3)^3 - \frac{1}{5}(3)^5 \right) - \left(3(0)^3 - \frac{1}{5}(0)^5 \right) \right\}$

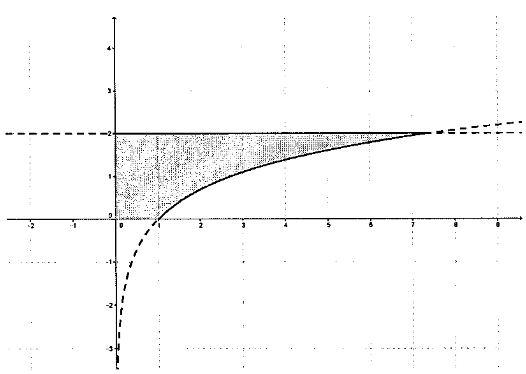
$= \frac{162\pi}{5}$

8. Find the volume obtained by rotating the region bounded by

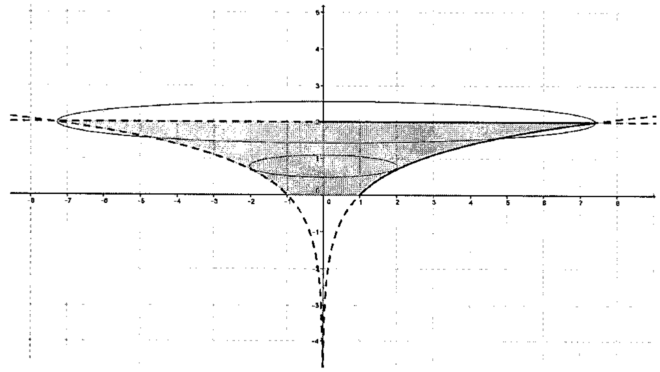
$$y = \ln x, \quad y = 2, \quad x = 0, \quad y = 0$$

about the y-axis

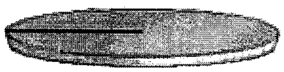
The region to be rotated looks like:



And after rotation, the solid looks like:



Cross sections perpendicular to the axis of rotation (the y-axis) are solid "disks".



The volume of a cylindrical disc is

$V = \pi R^2 h$

Where R is the radius and h is the thickness

In this case, the thickness is a small change in y, that is dy

The radius extends from $x = 0$ to $x =$ curve

Solving the curve for x:

$y = \ln x$

$e^y = e^{\ln x}$

$x = e^y$

So, $R = e^y$

The volume of the solid can be thought of as summing up the volume of each of these disks.

$$\int_{um}(\text{Volumes of disks})$$

$$\int_{um}\pi R^2 h$$

$$\int_{y=0}^{y=2}\pi\left(e^y\right)^2 dy$$

$$=\pi\int_0^2 e^{2y}dy$$

$$=\pi\int_0^4 e^u\left(\frac{du}{2}\right) \qquad \text{Let } u=2y \;\; ; \;\; y=0 \rightarrow u=0$$

$$=\frac{\pi}{2}\int_0^4 e^u du \qquad du=2dy \;\; ; \;\; y=2 \rightarrow u=4$$

$$=\frac{\pi}{2}\left\{\left(e^u\right)\Big|_0^4\right\} \qquad \frac{du}{2}=dy$$

$$=\frac{\pi}{2}\left\{\left(e^4\right)-\left(e^0\right)\right\}$$

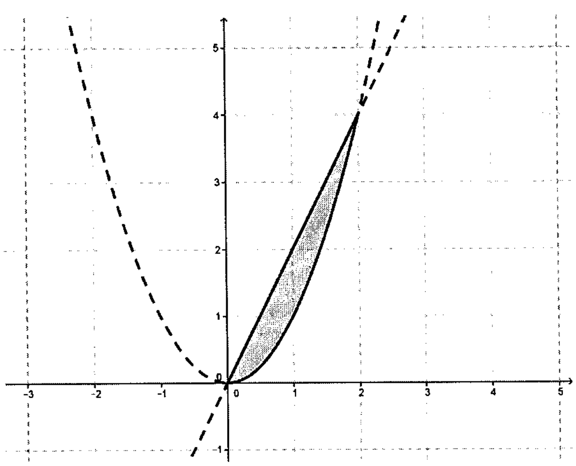
$$=\frac{\pi}{2}\left(e^4-1\right)$$

9. Use the shell method to determine the volume obtained by rotating the region bounded by

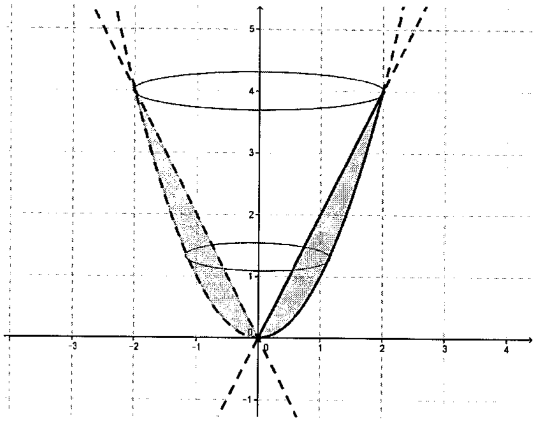
$$f(x)=x^2\ , \quad g(x)=2x$$

about the x-axis

The region to be rotated looks like:



And the solid formed by rotation looks like:



The shell method calculates the volume of the solid by considering it to be constructed by infinitely many nested "tubes" in this case arranged vertically

The sum of the volume of these shells determines the volume of the overall solid.

$$\int_{um} 2\pi(radius)(height)\,dx$$

$$= \int_0^2 2\pi(x)(g(x)-f(x))\,dx$$

$$= 2\pi\int_0^2 x(2x-x^2)\,dx$$

$$= 2\pi\int_0^2 (2x^2-x^3)\,dx$$

$$= 2\pi\left\{\left(\frac{2}{3}x^3-\frac{1}{4}x^4\right)\Big|_0^2\right\}$$

$$= 2\pi\left\{\left(\frac{2}{3}(2)^3-\frac{1}{4}(2)^4\right)-\left(\frac{2}{3}(0)^3-\frac{1}{4}(0)^4\right)\right\}$$

$$= \frac{8\pi}{3}$$

Integration by parts practice:

10. $\int x\cos 3x\,dx$

The integration by parts formula is: $\int u\ dv = vu - \int v\ du$

There are various mnemonics to help with how to choose which portions of the integrand to assign to the u variable and which parts to dv

One of my favorites is LIPET:

L = Logarithm functions

I = Inverse trig functions

P = Polynomial functions

E = Exponential functions

T = Trig functions

And the direction of variable choice goes this way:
$u \rightarrow \quad \leftarrow dv$
L I P E T
Following this recommendation, we would let u be the polynomial, and dv be the trigonometric function.

$$\int u\, dv = vu - \int v\, du \qquad \text{Let } u = x \;;\; dv = \cos 3x\, dx$$

$$\int (x)(\cos 3x\, dx) = \left(\frac{1}{3}\sin 3x\right)(x) - \int\left(\frac{1}{3}\sin 3x\right) dx \qquad \text{Then } du = dx \;;\; v = \frac{1}{3}\sin 3x$$

$$= \frac{1}{3}x\sin 3x - \frac{1}{3}\left(-\frac{1}{3}\cos 3x\right) + C$$

$$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$$

11. $\int \frac{\ln x}{x^3}\, dx$

Similarly, LIPET used on this problem recommends that u be the natural log and dv be the remaining rational function (not in the acronym).

$$\int u\, dv = vu - \int v\, du \qquad \text{Let } u = \ln x \;;\; dv = \frac{1}{x^3}\, dx$$

$$\int \frac{\ln x}{x^3}\, dx = \left(-\frac{1}{2}x^{-2}\right)(\ln x) - \int\left(-\frac{1}{2}x^{-2}\right)\left(\frac{1}{x}dx\right) \qquad \text{Then } du = \frac{1}{x}dx \;;\; v = -\frac{1}{2}x^{-2}$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\int x^{-3} dx$$

$$= -\frac{\ln x}{2x^2} + \frac{1}{2}\left(-\frac{1}{2}x^{-2}\right) + C$$

$$= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$$

12. $\int e^x \sin x\, dx$

Here, LIPET recommends that u be assigned to the exponential and dv be assigned to the trig function. (But we'll see that either assignment is fine, and the important part in this problem is to keep the same assignment when you have to do integration by parts the second time.)

$$\int u\, dv = vu - \int v\, du \qquad \text{Let } u = e^x \;;\; dv = \sin x\, dx$$

$$\int e^x \sin x\, dx = (-\cos x)(e^x) - \int(-\cos x)(e^x dx) \qquad \text{Then } du = e^x dx \;;\; v = -\cos x$$

$$= -e^x \cos x + \int e^x \cos x\, dx$$

$$= -e^x \cos x + \left\{(\sin x)(e^x) - \int(\sin x)(e^x dx)\right\} \qquad \text{Let } u = e^x \;;\; dv = \cos x\, dx$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx \qquad \text{Then } du = e^x dx \;;\; v = \sin x$$

At this point, we realize that the same integral as the original question, has shown up in our solution. Combining these, we have:

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

$$\int e^x \sin x\, dx + \int e^x \sin x\, dx = -e^x \cos x + e^x \sin x + C$$

$$2\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x + C$$

$$\int e^x \sin x\, dx = \frac{1}{2}\left(-e^x \cos x + e^x \sin x\right) + C$$

$$\int e^x \sin x\, dx = \frac{e^x}{2}\left(\sin x - \cos x\right) + C$$

Trigonometric Integrals practice:

13. $\int \sin^3 x \cos^2 x\, dx$

When you have powers of sine and cosine and if either of the powers is odd, generally, you can try to isolate one of terms that has the odd power:

$\int \sin^3 x \cos^2 x\, dx$

$= \int \sin x\left(\sin^2 x \cos^2 x\right) dx$

$= \int \sin x\left(\left(1 - \cos^2 x\right)\cos^2 x\right) dx$ Using the identity $\sin^2 x + \cos^2 x = 1$

$= \int \sin x\left(\cos^2 x - \cos^4 x\right) dx$ Let $u = \cos x$

$= \int \left(u^2 - u^4\right)\left(-du\right)$ $du = -\sin x\, dx$

$= \int \left(u^4 - u^2\right) du$ $-du = \sin x\, dx$

$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

14. $\int \cos^2 x \sin 2x \, dx$

When the arguments of the trig functions don't match, we need to look to trigonometric identities to try to find a substitution to make them match.

$\int \cos^2 x \sin 2x \, dx$ Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$

$= \int \cos^2 x (2\sin x \cos x) \, dx$

$= \int 2\sin x \cos^3 x \, dx$

$= 2\int (u)^3 (-du)$ Let $u = \cos x$

$= -2\int u^3 du$ $du = -\sin x \, dx$

$= \frac{-2}{4} u^4 + C$ $-du = \sin x \, dx$

$= -\frac{1}{2}\cos^4 x + C$

15. $\int \tan^3 x \sec x \, dx$

The derivative of $\tan x$ is $\sec^2 x$ and the derivative of $\sec x$ is $\sec x \tan x$

So, a simple *u*-substitution doesn't appear to solve this one. After separating the terms in a different way, and recalling the trig identity for $\tan^2 x$, then the pieces start to come together…

$\int \tan^3 x \sec x \, dx$

$= \int \tan x (\tan^2 x) \sec x \, dx$ Using the identity $\tan^2\theta = \sec^2\theta - 1$

$= \int \tan x (\sec^2 x - 1) \sec x \, dx$

$= \int (\sec^2 x - 1) \sec x \tan x \, dx$ Let $u = \sec x$

$= \int (u^2 - 1)(du)$ $du = \sec x \tan x \, dx$

$= \frac{1}{3}u^3 - u + C$

$= \frac{1}{3}\sec^3 x - \sec x + C$