M 328 K 58080 Midterm

1. Show that if a, b, c, d are integers and a divides b and c, then a divides $b^2 + cd + 2a$.

As $a|b$ we have that $b = ra$ for some integer r. Likewise, $c = sa$ for some integer s. So, $b^2 + cd + 2a = (ra)^2 + sad + 2a = (r^2a + sd + 2)a$ which shows that $b^2 + cd + 2a$ is a multiple of a.

2. Show that if a, b are positive integers with $a^3|b^2$ then $a|b$.

Write $a = p_1^{\alpha_1} \cdots p_m^{\alpha_m}, b = p_1^{\beta_1} \cdots p_m^{\beta_m}$, where the p_i are distinct primes and the α_i, β_i are non-negative integers. Then $a^3 = p_1^{3\alpha_1} \cdots p_m^{3\alpha_m}, b^2 = p_1^{2\beta_1} \cdots p_m^{2\beta_m}$ and, if $a^3|b^2$ we get $3\alpha_i \leq 2\beta_i, i = 1, \ldots, m$ by a corollary of unique factorization. Then, as $2/3 < 1$, we get $\alpha_i \leq (2/3)\beta_i \leq \beta_i, i = 1, \ldots, m$ and, consequently, $a|b$.

3. Find inverses of 2 and 3 modulo 101.

$101 = 2 \cdot 50 + 1$, so $2 \cdot (-50) \equiv 1 \pmod{101}$, so $-50 \equiv 51 \pmod{101}$ is an inverse of 2 modulo 101.

$101 = 3 \cdot 33 + 2$, $3 = 2 \cdot 1 + 1$, so $3 \cdot 34 \equiv 1 \pmod{101}$, so $34 \pmod{101}$ is an inverse of 2 modulo 101.

4. Prove that if an integer n is the square of another integer m (that is, $n = m^2$) then the least significant digit of the decimal representation of n is one of $0, 1, 4, 5, 6, 9$.

Using that if $m \equiv d \pmod{10}$, then $m^2 \equiv d^2 \pmod{10}$ and that m is congruent modulo 10 to one of $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, we get that $n = m^2$ is congruent modulo 10 to one of $0^2 = 0, 1^2 = 1, 2^2 = 2, 3^2 = 9, 4^2 = 16 \equiv 6 \pmod{10}$, $5^2 = 25 \equiv 5 \pmod{10}$,$6^2 = 36 \equiv 6 \pmod{10}$,$7^2 = 49 \equiv 9 \pmod{10}$, $8^2 = 64 \equiv 4 \pmod{10}$,$9^2 = 81 \equiv 1 \pmod{10}$. Removing duplicates leaves just $0, 1, 4, 5, 6, 9$.