1. Show that a rectangular box of a given volume has minimal surface area when the box is a cube.

Solution: Let the sides of the box be given by x, y, z and the volume be given by V. The surface area of the box is then $A(x,y) = 2(xy + yz + xz) = 2(xy + \frac{V}{x} + \frac{V}{y})$ where we used the restriction $z = \frac{V}{xy}$. The partial derivatives of A are then

$$A_x(x,y) = 2(y - \frac{V}{x^2})$$

$$A_y(x,y) = 2(x - \frac{V}{y^2}).$$

Setting both partial derivatives to 0 we get

$$y = \frac{V}{x^2} \implies x^2 y = V$$

$$x = \frac{V}{y^2} \implies y^2 x = V.$$

Dividing one equation by the other we conclude that $\frac{x}{y} = 1$ or equivalently $x = y$. Plugging it into the equation $x^2 y = V$ above we get

$$x^3 = V$$

and hence $x = y = V^{\frac{1}{3}}$. To see if it is a local minimum, maximum or a saddle point, we compute the discriminant. First compute second order partial derivatives of A

$$\begin{aligned} A_{xx} &= 4(\frac{V}{x^3}) \\ A_{yy} &= 4(\frac{V}{y^3}) \\ A_{xy} &= 1. \end{aligned}$$

Plugging in the critical point $(x_0, y_0) = \left(V^{\frac{1}{3}}, V^{\frac{1}{3}}\right)$, we get

$$A_{xx}(x_0, y_0) = 4$$

$$A_{yy}(x_0, y_0) = 4$$

$$A_{xy}(x_0, y_0) = 1$$

and the discriminant is

$$D = 16 - 1 = 15 > 0.$$

Hence (x_0, y_0) is either a local minimum or a local maximum. Since $A_{xx}(x_0, y_0) = 4 > 0$, it is a local minimum.

2. Find the point on the surface $y^2 = 9 - xz$ which is the closes to the point $(0, 0, 0)$.

Parametrize the surface by coordinates x, z:

$$r(x,z) = (x, \sqrt{9 - xz}, z).$$

To be exact, this only parametrizes half of the surface. There is also a sheet parametrized by $(x, -\sqrt{9-xz}, z)$. For a fixed x, z the points on the two sheets are at the same distance from $(0,0,0)$ so we will just focus on one of them. We will find (x, z) that will minimize the distance squared, i,e.

$$f(x,z) = |r(x,z)|^2 = x^2 + 9 - xz + z^2.$$

Taking partial derivatives we get

$$f_x = 2x - z$$

$$f_z = -x + 2z.$$

Setting both to 0 we get

$$z = 2x$$

$$x = 2z.$$

The only solution is $x = z = 0$. We get a critical point of f at $(0,0)$. Which corresponds to point $(0,3,0)$ on the surface. Taking partial derivatives we get

$$\begin{aligned} f_{xx} &= 2 \\ f_{zz} &= 2 \\ f_{zx} &= -1. \end{aligned}$$

By the discriminant method, $(0,0)$ is a local minimum. So the points $(0,3,0)$ and $(0,-3,0)$ are the closest to the origin.

3. Find the equation of the plane that passes through the point $(1,2,3)$ and cuts off the smallest volume in the first quadrant.

The equation of a plane P passing through the point $(1,2,3)$ is given by

$$A(x-1) + B(y-2) + C(z-3) = 0.$$

The volume that is being cut off is a triangular pyramid. The volume of a pyramid is $\frac{1}{3}Ah$ where A is the area of the base and h is the height. In particular, if x_0, y_0, z_0 are the intersections of the plane P with the x, y, z axes, then the volume of the pyramid is $\frac{x_0 y_0 z_0}{6}$ (notice that you only need to know that the volume is proportional to $x_0 y_0 z_0$ to solve this problem). In our case, the intersects are

$$\begin{aligned} x_0 &= \frac{A + 2B + 3C}{A} \\ y_0 &= \frac{A + 2B + 3C}{B} \\ z_0 &= \frac{A + 2B + 3C}{C} \end{aligned}$$

and the volume of the pyramid is given by

$$6V = \frac{(A + 2B + 3C)^3}{ABC}.$$

Given a plane P, the numbers A, B, C are only defined up to multiplication by a constant which we can now fix to make the computation simpler. Let's fix $ABC = 1$ and re-express $A = \frac{1}{BC}$. We get

$$6V = \left(\frac{1}{BC} + 2B + 3C\right)^3.$$

We are interested in minimizing V. It is equivalent to minimizing $V'(B, C) = 6 \cdot \sqrt[3]{V} = \left(\frac{1}{BC} + 2B + 3C\right)$. We find

$$V'_B = 2 - \frac{1}{B^2C}$$

$$V'_C = 3 - \frac{1}{BC^2}.$$

Setting these to 0 we get

$$B^2C = \frac{1}{2}$$

$$C^2B = \frac{1}{3}$$

Dividing one by the other we get

$$\frac{C}{B} = \frac{2}{3}$$

and hence

$$B^2(\frac{2B}{3}) = \frac{1}{2} \Longrightarrow B = \sqrt[3]{\frac{3}{4}}$$

$$C = \frac{2}{3}\sqrt[3]{\frac{3}{4}}$$

$$A = \frac{1}{\frac{2}{3}\sqrt[3]{\frac{3}{4}} \cdot \sqrt[3]{\frac{3}{4}}} = \frac{\sqrt[3]{\frac{3}{4}}}{\frac{2}{3} \cdot \frac{3}{4}} = 2 \cdot \sqrt[3]{\frac{3}{4}}.$$

We now recall that A, B, C can be multiplied by a constant and still define the same surface. In particular multiplying by $\frac{3}{\sqrt[3]{\frac{3}{4}}}$ we get the plane given by

$$6(x - 1) + 3(y - 2) + 2(z - 3) = 0.$$