DYNAMICAL PROPERTIES OF CONVEX COCOMPACT ACTIONS IN PROJECTIVE SPACE

THEODORE WEISMAN

Abstract. We give a dynamical characterization of convex cocompact group actions on properly convex domains in projective space in the sense of Danciger-Guéritaud-Kassel: we show that convex cocompactness in $\mathbb{RP}^d$ is equivalent to an expansion property of the group about its limit set, occuring in different Grassmannians. As an application, we give a sufficient and necessary condition for convex cocompactness for groups which are hyperbolic relative to a collection of convex cocompact subgroups. We show that convex cocompactness in this situation is equivalent to the existence of an equivariant homeomorphism from the Bowditch boundary to the quotient of the limit set of the group by the limit sets of its peripheral subgroups.

Contents

1. Introduction

When G is a rank-one semisimple Lie group, a *convex cocompact* subgroup is a discrete group $\Gamma \subset G$ which acts cocompactly on some convex set in the Riemannian symmetric space G/K, where K is a maximal compact in G. Convex cocompact subgroups in rank-one have long been objects of great interest, and have a wide variety of possible characterizations.

More recently, efforts have been underway to understand the appropriate generalization of convex cocompactness in higher-rank Lie groups. A key concept is *Anosov representations*: discrete and faithful representations of word-hyperbolic groups into reductive Lie groups which generalize a *dynamical* definition of convex cocompact subgroups in rank one. Anosov representations were first defined for surface groups by Labourie [Lab06], and the definition was later extended to general word-hyperbolic groups by Guichard-Wienhard [GW12]. Guichard-Wienhard

Date: June 29, 2023.

also demonstrated that an Anosov representation ρ of a group Γ can be interpreted as the holonomy of a certain geometric structure associated to ρ. Currently, understanding the connection between Anosov representations and geometric structures is an area of active research.

In [DGK17], Danciger, Guéritaud, and Kassel developed a notion of convex cocompact representations in $\mathrm{PGL}(d,\mathbb{R})$ that (as in the rank-one setting) have concrete and transparent *convex* geometric objects associated to them—in this case, a compact manifold (or orbifold) with a *convex projective structure.*

Recall that a subset Ω of projective space $\mathbb{RP}^{d-1}$ is *convex* if it is contained in some affine chart in $\mathbb{RP}^{d-1}$, and Ω is a convex subset of that affine chart. The set Ω is *properly convex* if its closure is also contained in an affine chart, and it is a *properly convex domain* if it is also open. A *convex projective orbifold* is a quotient of a convex set in $\mathbb{RP}^{d-1}$ by a discrete subgroup of $\mathrm{Aut}(\Omega)$, where

$$\mathrm{Aut}(\Omega) := \{g \in \mathrm{PGL}(d,\mathbb{R}) : g \cdot \Omega = \Omega\}.$$

The Danciger-Guéritaud-Kassel definition of convex cocompactness in $\mathbb{RP}^{d-1}$ says that a group $\Gamma \subset \mathrm{PGL}(d,\mathbb{R})$ is convex cocompact when it is the holonomy of a compact convex projective orbifold satisfying certain conditions.

Definition 1.1. Let Ω be properly convex domain in $\mathbb{RP}^{d-1}$, and let $\Gamma \subseteq \mathrm{Aut}(\Omega)$.

- The *full orbital limit set* $\Lambda_\Omega(\Gamma)$ is the set of accumulation points in $\partial\Omega$ of Γ-orbits in Ω, i.e. the union
$$\bigcup_{x\in\Omega}(\overline{\Gamma\cdot x}\cap\partial\Omega).$$
- The *convex core* of Γ in Ω, denoted $\mathrm{Cor}_\Omega(\Gamma)$, is the convex hull in Ω of the full orbital limit set $\Lambda_\Omega(\Gamma)$.

Definition 1.2 ([DGK17, Definition 1.11])**.** Let Ω be a properly convex domain in $\mathbb{RP}^{d-1}$, and let Γ be a discrete group acting by projective transformations on Ω. The group Γ *acts convex cocompactly on* Ω if it acts cocompactly on $\mathrm{Cor}_\Omega(\Gamma)$.

A group $\Gamma \subset \mathrm{PGL}(d,\mathbb{R})$ *acts convex cocompactly in* $\mathbb{RP}^{d-1}$ if it acts convex cocompactly on *some* properly convex domain $\Omega \subset \mathbb{RP}^{d-1}$.

Note that this definition is strictly stronger than merely asking for Γ to act cocompactly on some Γ-invariant convex subset of a properly convex domain Ω.

Danciger-Guéritaud-Kassel prove that when a discrete word-hyperbolic group $\Gamma \subset \mathrm{PGL}(d,\mathbb{R})$ preserves a properly convex domain, Γ acts convex cocompactly on some domain $\Omega \subset \mathbb{RP}^{d-1}$ precisely when the inclusion $\Gamma \hookrightarrow \mathrm{PGL}(d,\mathbb{R})$ is P_1-*Anosov*; a related result was independently shown by Zimmer in [Zim21]. Moreover, a group acting convex cocompactly on a domain Ω is word-hyperbolic precisely when there are no nontrivial projective segments in its full orbital limit set (and in this case the definition is equivalent to a similar notion of convex cocompactness introduced by Crampon-Marquis in [CM14]).

However, there are also many examples of non-hyperbolic groups which have convex cocompact representations as in Definition 1.2; see Section 2.6. These non-hyperbolic convex cocompact groups are somewhat more mysterious than their hyperbolic counterparts. Recently, however, there has been significant progress towards a deeper understanding of them, especially in the case where the Γ-action on the entire domain Ω is cocompact: see e.g. [Isl19], [Bob20], [Zim20]. Of particular relevance to this paper is the description, due to Islam-Zimmer [IZ21], [IZ19], of the domains with a convex cocompact action by a relatively hyperbolic group relative to a family of virtually abelian subgroups of rank at least two.

1.1. **Convex cocompactness and Anosov dynamics.** The first main result of this paper (Theorem 1.5 below) is a dynamical characterization of convex cocompactness that applies even for non-hyperbolic groups, generalizing the relationship between convex cocompactness and P_1-Anosov representations. This addresses a question asked by Danciger-Guéritaud-Kassel in [DGK17].

The main idea behind Theorem 1.5 is to generalize the dynamical description of convex cocompactness explored by Sullivan [Sul85] in the rank-one setting: when Γ is a discrete subgroup of $\mathrm{PO}(d,1)$, Γ is convex cocompact if and only if Γ satisfies an expansion/contraction property about its limit set in $\partial\mathbb{H}^d$.

More generally, when $\rho : \Gamma \to G$ is a P-Anosov representation (for P a parabolic subgroup of a reductive Lie group G), work of Kapovich-Leeb-Porti [KLP17] shows that Γ satisfies an expansion property on the flag manifold G/P. Kapovich-Leeb-Porti also showed in [KLP18] (using the same basic idea as Sullivan) that this expansion property can be used to find cocompact domains of discontinuity for Anosov representations in G/P.

When Γ is hyperbolic with $\rho : \Gamma \to \mathrm{PGL}(d,\mathbb{R})$ convex cocompact, ρ is P_1-Anosov, yielding an expansion property in $\mathbb{RP}^{d-1}$. Theorem 1.5 says that convex cocompact representations $\Gamma \to \mathrm{PGL}(d,\mathbb{R})$ are characterized by a similar expansion property on *multiple* flag manifolds (and some additional technical conditions): different elements of Γ expand neighborhoods in different Grassmannians $\mathrm{Gr}(k,d)$. That is, convex cocompactness in $\mathrm{PGL}(d,\mathbb{R})$ is equivalent to a kind of "mixed Anosov" property.

Using work of Cooper-Long-Tillmann [CLT18], Danciger-Guéritaud-Kassel also observe that convex cocompactness in the sense of Definition 1.2 is *stable*: if $\Gamma \subset \mathrm{PGL}(d,\mathbb{R})$ is a convex cocompact subgroup, then there is an open subset $\mathcal{U}$ of $\mathrm{Hom}(\Gamma, \mathrm{PGL}(d,\mathbb{R}))$, containing the inclusion $\Gamma \hookrightarrow \mathrm{PGL}(d,\mathbb{R})$, such that any $\rho \in \mathcal{U}$ is injective and discrete with $\rho(\Gamma)$ convex cocompact.

Theorem 1.5 then implies that the "mixed Anosov" property we consider in this paper is also stable under small deformations. This suggests one possible route towards a generalization of Anosov representations for non-hyperbolic groups.

1.1.1. *Structure of the boundary of a convex domain.* The expansion property we use to characterize convex cocompactness in $\mathrm{PGL}(d,\mathbb{R})$ is given in terms of the natural decomposition of the boundary of a convex domain into convex pieces. Each point x in the boundary of a convex set Ω lies in a unique *face* $F_\Omega(x)$: the set of all points in $\partial\Omega$ which lie in a common open line segment in $\partial\Omega$ with x. The *support* $\mathrm{supp}(F)$ of a face F in $\partial\Omega$ is the projective span of F. We can view the support as an element of the Grassmannian of k-planes $\mathrm{Gr}(k,d)$, for some $1 \leq k < d$.

When Λ is a subset of $\partial\Omega$, we can ask for it to be well-behaved with respect to the decomposition of $\partial\Omega$ into faces.

Definition 1.3. Let Λ be a subset of $\partial\Omega$. If, for all $x \in \Lambda$, we have

$$F_\Omega(x) \subset \Lambda,$$

we say that Λ *contains all of its faces.*

Definition 1.4. Let Λ be a subset of $\partial\Omega$. We say that Λ is *boundary-convex* if any supporting hyperplane of Ω intersects Λ in a convex set.

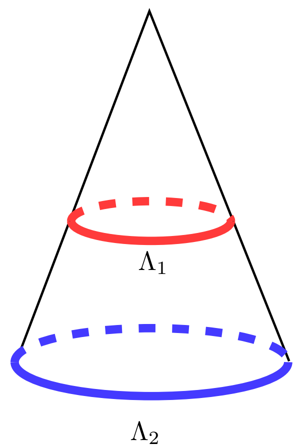

FIGURE 1. Here Ω is a cone, the convex hull of a disk and a point in $\mathbb{R}\mathrm{P}^3$. The set Λ_1 (red) does not contain all of its faces (Definition 1.3), because each point in Λ_1 is contained in the interior of a line segment in $\partial\Omega$ which is not contained in Λ_1. The set Λ_2 (blue) contains all of its faces, but it is not boundary-convex (Definition 1.4): a line segment joining two points of Λ_2 intersects $\partial\Omega - \Lambda_2$.

If $\Lambda = \Lambda_\Omega(\Gamma)$ for a group $\Gamma \subseteq \mathrm{Aut}(\Omega)$ acting convex cocompactly on some domain Ω, then [DGK17, Lemma 4.1] implies that Λ is closed and boundary convex, and contains all of its faces.

When Λ is a subset of $\partial\Omega$ containing all of its faces, and $\Gamma \subset \mathrm{PGL}(d, \mathbb{R})$ is a group preserving Λ, we say Γ is *expanding at the faces of* Λ (Definition 3.1) if for every face F in Λ, the group Γ has an element which is expanding in a neighborhood of $\mathrm{supp}(F)$ in $\mathrm{Gr}(k, d)$. When the expansion constants can be chosen uniformly, we say Γ is *uniformly* expanding at the faces of Λ. For the full definitions, see Section 3.1.

Here is the precise version of our characterization of convex cocompactness:

Theorem 1.5. *Let Ω be a properly convex domain in $\mathbb{R}\mathrm{P}^{d-1}$, and let Γ be a discrete subgroup of $\mathrm{Aut}(\Omega)$. The following are equivalent:*

(1) Γ acts convex cocompactly on Ω.
(2) There is a closed, Γ-invariant, and boundary-convex subset $\Lambda \subset \partial\Omega$ with nonempty convex hull such that Λ contains all of its faces and Γ is uniformly expanding at the faces of Λ.

In this case, the set Λ is the full orbital limit set $\Lambda_\Omega(\Gamma)$.

Remark 1.6. When we prove the implication (2) $\implies$ (1) of Theorem 1.5, we will not actually need to assume that the expansion at the faces of Λ is uniform—only that the expansion occurs with respect to a particular choice of Riemannian metric on $\mathrm{Gr}(k, d)$. See Remark 3.2.

A special case of convex cocompactness is when a discrete group $\Gamma \subset \mathrm{Aut}(\Omega)$ acts cocompactly on all of Ω. In this case, we say that Ω is *divisible*, and the group Γ *divides* the domain. As $\partial\Omega$ is always boundary convex and contains all of its faces, when $\Lambda = \partial\Omega$, Theorem 1.5 can be stated as the following:

Corollary 1.7. *Let Γ be a discrete subgroup of $\mathrm{PGL}(d, \mathbb{R})$ preserving a properly convex domain Ω. Then Γ divides Ω if and only if Γ is uniformly expanding at the faces of $\partial\Omega$.*

1.1.2. *Spaces of properly convex domains.* A key technical tool we need for the implication $2 \implies 1$ in Theorem 1.5 is a version of the *Benzécri cocompactness theorem* for properly convex domains in $\mathbb{RP}^{d-1}$, which applies relative to a direct sum decomposition of $\mathbb{R}^d$. This result (proved in Section 4) may be of independent interest, so we describe it briefly here.

In [Ben60], Benzécri showed that the group $\mathrm{PGL}(d, \mathbb{R})$ acts properly and cocompactly on the space of pointed properly convex domains in $\mathbb{RP}^{d-1}$. One immediate and useful consequence of this fact is that if $\Omega \subset \mathbb{RP}^{d-1}$ is a properly convex domain, and x_n is any sequence of points in Ω, then after extracting a subsequence, there is a sequence of projective transformations $g_n \in \mathrm{PGL}(d, \mathbb{R})$ such that $g_n\Omega$ converges to a fixed properly convex domain Ω_∞, and $g_n x_n$ converges to a point in the interior of Ω_∞.

We prove a result (Proposition 4.4) which provides some control over the group elements g_n appearing above. Roughly, our result says that if the sequence x_n lies in a lower-dimensional "slice" $W \subset \Omega$ satisfying some technical conditions, then the sequence g_n above can be chosen to preserve *both* the projective subspace spanned by W and a fixed complementary subspace in $\mathbb{R}^d$.

Proposition 4.4 can be compared to earlier work of Benoist [Ben03] and Frankel [Fra91]. These results effectively show that when x_n lies in a lower-dimensional "slice" in Ω as above, then the sequence g_n can always be chosen to preserve the span of the slice. The proposition we prove in this paper has stronger hypotheses, but also a stronger conclusion. The more precise control we get is necessary for the intended application—see Remark 5.11.

1.2. **Relative hyperbolicity.** In the second part of this paper, we use the dynamical characterization of convex cocompactness given by Theorem 1.5 to study convex cocompactness for a group Γ which is hyperbolic relative to a collection $\mathcal{H}$ of convex cocompact subgroups. We will give a necessary and sufficient condition for such a group to act convex cocompactly in terms of an embedding of the Bowditch boundary $\partial(\Gamma, \mathcal{H})$. This strengthens the connection between convex cocompact groups in $\mathrm{PGL}(d, \mathbb{R})$ and Anosov representations, since Anosov representations can also be characterized by the existence of an equivariant embedding of the *Gromov* boundary of a hyperbolic group into some flag manifold (see [GW12], [KLP17]).

We will give a definition of relatively hyperbolic groups in terms of convergence dynamics in Section 6.

Definition 1.8. Let $\mathcal{H} = \{H_i\}$ be a collection of subgroups of $\mathrm{PGL}(d, \mathbb{R})$, each acting convex cocompactly on a fixed properly convex domain Ω with pairwise disjoint full orbital limit sets $\Lambda_\Omega(H_i)$.

We denote by $[\partial\Omega]_{\mathcal{H}}$ the space obtained from $\partial\Omega$ by collapsing all of the full orbital sets $\Lambda_\Omega(H_i)$ to points. Similarly, for $x \in \partial\Omega$, or a subset $\Lambda \subseteq \partial\Omega$, we use $[x]_{\mathcal{H}}$ and $[\Lambda]_{\mathcal{H}}$ to denote the images of x and Λ in $[\partial\Omega]_{\mathcal{H}}$.

When $\mathcal{H}$ is a conjugacy-invariant collection of subgroups of a group $\Gamma \subseteq \mathrm{Aut}(\Omega)$, the action of Γ on $\partial\Omega$ descends to an action on $[\partial\Omega]_{\mathcal{H}}$. More generally, if $\Lambda \subseteq \partial\Omega$ is Γ-invariant, Γ also acts on $[\Lambda]_{\mathcal{H}}$.

We show the following:

Theorem 1.9. *Let $\Gamma \subseteq \mathrm{PGL}(d, \mathbb{R})$ act on a properly convex domain Ω, and suppose that Γ is hyperbolic relative to a family of subgroups $\mathcal{H} = \{H_i\}$, such that the H_i each act convex cocompactly on Ω with pairwise disjoint full orbital limit sets.*

If there is a boundary-convex Γ-invariant subset $\Lambda \subseteq \partial\Omega$ containing all of its faces, and a Γ-equivariant embedding $\partial(\Gamma, \mathcal{H}) \to [\partial\Omega]_{\mathcal{H}}$ with image $[\Lambda]_{\mathcal{H}}$, then Γ acts convex cocompactly on Ω and Λ is the full orbital limit set $\Lambda_\Omega(\Gamma)$.

Remark 1.10. In Theorem 1.9, we do not need to assume that Γ is discrete in $\mathrm{PGL}(d, \mathbb{R})$: this will also follow from the existence of the equivariant boundary embedding.

There are two special cases of Theorem 1.9 worth considering, which we state separately as corollaries. The first is when the subset Λ is the entire boundary $\partial\Omega$.

Corollary 1.11. *Let Γ, Ω, and $\mathcal{H}$ be as in Theorem 1.9, and suppose that $\partial(\Gamma, \mathcal{H})$ is equivariantly homeomorphic to $[\partial\Omega]_{\mathcal{H}}$. Then Γ divides Ω.*

The second corollary is when the set of peripheral subgroups is empty, i.e. Γ is hyperbolic.

Corollary 1.12. *Let Γ be a word-hyperbolic group in $\mathrm{PGL}(d, \mathbb{R})$ acting on a properly convex domain Ω, and suppose that the Gromov boundary of Γ embeds equivariantly into $\partial\Omega$ with image Λ.*

If Λ is boundary-convex and contains all of its faces, then Γ acts convex cocompactly on Ω and $\Lambda = \Lambda_\Omega(\Gamma)$.

When a hyperbolic group acts convex cocompactly on a domain Ω, its full orbital limit set contains no segments. So in this case, Λ contains all of its faces whenever no point of Λ lies in the interior of any segment in $\partial\Omega$.

We also can phrase this corollary in terms of Anosov boundary maps. Due to [DGK17, Theorem 1.15] (see also [Zim21, Theorem 1.10]), if a word-hyperbolic group Γ acts convex cocompactly on some domain Ω, then the inclusion map $\Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ is a P_1-Anosov representation preserving Ω, and in this case the full orbital limit set is the image of the Anosov boundary map $\partial\Gamma \to \mathbb{R}\mathrm{P}^{d-1}$. Thus Corollary 1.12 implies:

Corollary 1.13. *Let Γ be a word-hyperbolic subgroup of $\mathrm{PGL}(d, \mathbb{R})$ preserving a properly convex domain Ω, and suppose that there exists a Γ-equivariant embedding $\xi : \partial\Gamma \to \partial\Omega$ whose image is boundary-convex and contains all of its faces. Then the inclusion $\Gamma \hookrightarrow \mathrm{PGL}(d, \mathbb{R})$ is a P_1-Anosov representation with $\mathbb{R}\mathrm{P}^{d-1}$ boundary map ξ.*

Note that it is not true in general that the $\mathbb{R}\mathrm{P}^{d-1}$-boundary map ξ of a P_1-Anosov representation always embeds into the boundary of some properly convex domain $\Omega \subset \mathbb{R}\mathrm{P}^{d-1}$. Moreover even if ξ does embed into $\partial\Omega$ for some Ω, it does not necessarily follow that the image of the embedding is boundary-convex. However, it again follows from [DGK17, Theorem 1.15] that in this case, there is *some* Ω' such that ξ embeds $\partial\Gamma$ into $\partial\Omega'$ with a boundary-convex image containing its faces. In fact it is always possible to take Ω' strictly convex with C^1 boundary.

Remark 1.14. Kapovich-Leeb [KL18] and Zhu [Zhu21] (see also Zhu-Zimmer [ZZ22]) have given several possible definitions for a relative Anosov representation of a relatively hyperbolic group, aiming to generalize geometrical finiteness in rank-one in the same way that Anosov representations generalize convex cocompactness.

The non-hyperbolic convex cocompact group actions we consider in this paper are not covered by either the Kapovich-Leeb or Zhu pictures. For one, not all examples of convex cocompact groups are relatively hyperbolic (see Section 2.6). But even in the relatively hyperbolic setting, the definitions are not compatible. Due to [DGK17, Proposition 10.3], convex cocompact groups in $\mathrm{PGL}(d,\mathbb{R})$ do not contain weakly unipotent elements. In contrast, the relative Anosov subgroups considered by Kapovich-Leeb and Zhu always contain weakly unipotent elements if the group is not hyperbolic (see section 5 in [KL18]). However, see [Wei22] for follow-up work which gives a unified approach to studying both relatively hyperbolic convex cocompact groups in $\mathrm{PGL}(d,\mathbb{R})$ and relative Anosov representations.

During the proof of Theorem 1.9, we will see the following (see Proposition 7.1):

Proposition 1.15. *In the setting of Theorem 1.9, every nontrivial segment in the set Λ is contained in the full orbital limit set of some $H_i \in \mathcal{H}$.*

This leads us to a converse to Theorem 1.9.

Theorem 1.16. *Let Γ be a group acting convex cocompactly on a properly convex domain Ω, and suppose that Γ has a conjugacy-invariant collection of subgroups $\mathcal{H} = \{H_i\}$, such that the groups in $\mathcal{H}$ lie in finitely many conjugacy classes and each H_i acts convex cocompactly on Ω.*

Then Γ is hyperbolic relative to $\mathcal{H}$ if and only if

(i) the full orbital limit sets $\Lambda_\Omega(H_i)$, $\Lambda_\Omega(H_j)$ are disjoint for distinct $H_i, H_j \in \mathcal{H}$,

(ii) every nontrivial segment in $\Lambda_\Omega(\Gamma)$ is contained in $\Lambda_\Omega(H_i)$ for some $H_i \in \mathcal{H}$, and

(iii) each $H_i \in \mathcal{H}$ is its own normalizer in Γ.

Moreover, in this case, $\partial(\Gamma, \mathcal{H})$ equivariantly embeds into $[\partial\Omega]_{\mathcal{H}}$ with image $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.

Remark 1.17. If conditions (i) and (ii) hold for a conjugacy-closed collection of subgroups $\mathcal{H}$ of Γ, then they also hold for the collection of normalizers, since $g \cdot \Lambda_\Omega(H_i) = \Lambda_\Omega(gH_ig^{-1})$ for any $H_i \in \mathcal{H}$.

Moreover, condition (iii) of Theorem 1.16 is always true for the peripheral subgroups of a relatively hyperbolic group, because then each $H_i \in \mathcal{H}$ can be exactly realized as the stabilizer of its unique fixed point in the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ (see Theorem 6.3).

Islam and Zimmer [IZ21, IZ19] have previously shown that when Γ is a convex cocompact group which is hyperbolic relative to a collection $\mathcal{H}$ of virtually abelian subgroups of rank at least 2, conditions (i) and (ii) of Theorem 1.16 hold, and moreover the assumption that all of the groups in $\mathcal{H}$ act convex cocompactly is automatically satisfied. In particular, this implies that the set $[\partial\Omega]_{\mathcal{H}}$ is well-defined.

Thus, in this case, Theorem 1.16 implies the following:

Corollary 1.18. *Let Ω be a properly convex domain, and let Γ be a group which is hyperbolic relative to a collection $\mathcal{H}$ of virtually abelian subgroups with rank at least 2.*

If Γ acts convex cocompactly on Ω, then there is an equivariant embedding from $\partial(\Gamma, \mathcal{H})$ to $[\partial\Omega]_{\mathcal{H}}$ whose image is $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.

Islam-Zimmer have recently shown in [IZ20] that if Γ acts convex cocompactly on some domain Ω, and Γ is isomorphic to the fundamental group of an orientable, closed, and irreducible 3-manifold M which is non-geometric (i.e. does not carry one of the eight Thurston geometries), then M decomposes into hyperbolic pieces and therefore Γ is hyperbolic relative to free abelian subgroups. Using different methods, they also give a proof of Corollary 1.18 in this case.

Theorem 1.16 in fact applies to a strictly larger family of groups than those covered by Corollary 1.18: there exist relatively hyperbolic convex cocompact groups which are *not* hyperbolic relative to virtually abelian subgroups (and are not themselves hyperbolic). See the end of Section 2.6.3 for details.

Remark 1.19. Since this paper first appeared, Islam-Zimmer [IZ22] have shown that when Γ is *any* relatively hyperbolic group acting convex cocompactly on a properly convex domain Ω, then each peripheral subgroup of Γ *always* acts convex cocompactly on Ω as well. This makes it possible to drop an assumption from Theorem 1.16, and implies that Corollary 1.18 holds in the case of *arbitrary* peripheral subgroups.

In the same paper, Islam-Zimmer extend some of our techniques to show versions of Theorem 1.16 and Corollary 1.18 in the more general context of *naive* convex cocompact group actions in projective space, which we do not discuss in this paper.

Remark 1.20. Relatively hyperbolic group actions on convex projective domains have previously been studied by Crampon-Marquis [CLM16] (who provide a notion of a geometrically finite group action on a strictly convex domain), and by Cooper-Long-Tillmann [CLT15] in the context of convex projective cusps. Choi [Cho10] has also studied the relationship between the projective geometry of strictly convex projective orbifolds with ends, and the Bowditch boundaries of their (relatively hyperbolic) fundamental groups.

The relatively hyperbolic group actions we consider in this paper are different, however, since we consider projective orbifolds which are *not* necessarily strictly convex. In this context, the peripheral subgroups do *not* need to be associated to an end of the manifold, and act *cocompactly* on a convex subset of projective space.

1.3. **Outline of the paper.** In Section 2, we recall some background about properly convex domains and their automorphism groups, and describe some known examples of convex cocompact groups.

Sections 3 through 5 of this paper are devoted to the proof of Theorem 1.5. We first prove the implication (2) $\implies$ (1) in Section 3, using a modified version of an analogous argument in [DGK17]. The proof proceeds by contradiction: we assume that the space of Γ-orbits is not compact in $\mathrm{Cor}_\Omega(\Gamma)$, which means that the orbits uniformly accumulate on the boundary of $\mathrm{Cor}_\Omega(\Gamma)$ in Ω. But the expansivity of the action on the faces of Λ allows us to move points away from the boundary of $\mathrm{Cor}_\Omega(\Gamma)$ and we get a contradiction. This basic idea is essentially due to Sullivan [Sul85], but here the argument is more complicated because the expansivity occurs in a different space than the one where we want to find a cocompact action.

Remark 1.21. In [KLP18], Kapovich-Leeb-Porti also adapt Sullivan's argument to show that an expanding (Anosov) action in one flag manifold can produce a cocompact domain of discontinuity in another flag manifold. Their argument does not apply directly to our situation, since it relies on the fact that the "limit set" where the expanding action occurs is *compact*. Typically, if Λ is the full orbital limit set of a group acting convex cocompactly on $\Omega \subset \mathbb{RP}^{d-1}$, the subset of $\mathrm{Gr}(k, d)$ consisting of supports of k-dimensional faces of the full orbital limit set Λ is *not* compact (see e.g. [Ben06a] for examples). Consequently, it is necessary for us to work with expansion in several different Grassmannians for the same group—and to exploit the fact that the full orbital limit set Λ actually lies in the boundary of a convex subset of $\mathbb{RP}^{d-1}$.

Section 4 of the paper is mainly devoted to the proof of Proposition 4.4, the key technical result mentioned earlier in 1.1.2. Then, in Section 5, we use the results of Section 4 to prove the implication (1) $\implies$ (2) of Theorem 1.5. The basic argument is again inspired by Sullivan's work [Sul79] in the case of convex cocompact actions in $\mathbb{H}^d$.

The remainder of the paper (Sections 6 through 8) restricts to the context of relatively hyperbolic groups. In Section 6, we give some background on relatively hyperbolic groups, and state a dynamical definition of relatively hyperbolic groups due to Yaman [Yam04]. Then in Section 7, we use this characterization to prove Theorem 1.9, relying on the characterization of convex cocompactness given by Theorem 1.5.

In Section 8, we extend a result of Islam-Zimmer [IZ19], showing that when Γ is a group acting on a properly convex domain Ω, and Γ is hyperbolic relative to a collection of subgroups acting convex cocompactly on Ω, conditions (i) and (ii) of Theorem 1.16 hold. Then, we prove the rest of Theorem 1.16, again by applying Theorem 1.5 and using the dynamical definition of relative hyperbolicity from Section 6.

1.4. **Acknowledgments.** The author would like to thank his advisor, Jeff Danciger, for providing endless amounts of guidance and encouragement. We also would like to thank Fanny Kassel for giving useful feedback, and two anonymous referees for helpful comments and insights.

2. Background on properly convex domains

2.1. **Basic definitions.** All real vector spaces in this paper are finite-dimensional.

2.1.1. *Convex cones and convex domains.*

Definition 2.1. A *convex cone* in a real vector space V is a convex subset of $V - \{0\}$ which is closed under multiplication by positive scalars.

A convex cone is *sharp* if it does not contain any affine line.

The *boundary* of a convex cone C in a real vector space V is the boundary of C viewed as a cone in its linear span V' in V; this boundary is homeomorphic to a cone over S^{k-2}, where $k = \dim V'$.

Definition 2.2. A subset $\Omega \subset \mathbb{P}(V)$ is *convex* if it is the projectivization of some convex cone $\tilde{\Omega} \subset V$, and it is *properly convex* if $\tilde{\Omega}$ is sharp (equivalently, if $\overline{\Omega}$ is contained in some affine chart in $\mathbb{P}(V)$). An open convex set is a *convex domain.*

The boundary $\partial\Omega$ is the projectivization of $\partial\tilde{\Omega} - \{0\}$. A convex set Ω is *strictly convex* if $\partial\Omega$ does not contain a nontrivial projective segment.

Definition 2.3. Let Ω be a convex subset of $\mathbb{P}(V)$. A *supporting subspace* of Ω is a projective subspace which intersects $\partial\Omega$ but not Ω. In particular, a *supporting hyperplane* is a codimension-1 supporting subspace.

Using the convex cone over Ω, one can easily verify the following:

Proposition 2.4. *Let Ω be a convex subset of $\mathbb{P}(V)$. Every point $x \in \partial\Omega$ is contained in at least one supporting hyperplane.*

We remark that a convex domain in $\mathbb{P}(V)$ has C^1 boundary precisely when every point in $\partial\Omega$ is contained in *exactly* one supporting hyperplane.

2.1.2. *Projective line segments.* When Ω is a properly convex set, and $x, y \in \overline{\Omega}$, we use $[x, y]$ to denote the unique (closed) projective line segment joining x and y which is contained in $\overline{\Omega}$. We similarly use (x, y), $[x, y)$, $(x, y]$ to denote open and half-open projective line segments.

2.1.3. *Convex hull and ideal boundary.*

Definition 2.5. If Ω is a properly convex set in $\mathbb{P}(V)$ and $\Lambda \subset \partial\Omega$, then the *convex hull* of Λ is its convex hull in Ω in any affine chart containing Ω. We denote the convex hull of Λ by $\mathrm{Hull}_\Omega(\Lambda)$.

The *ideal boundary* of a set C in a properly convex set Ω is the set

$$\partial_i(C) := \overline{C} \cap \partial\Omega,$$

where the closure of C is taken in $\mathbb{P}(V)$.

If Λ is a subset of $\partial\Omega$ with nonempty convex hull, Λ is *boundary-convex* (Definition 1.4) precisely when

$$\Lambda = \partial_i \mathrm{Hull}_\Omega(\Lambda).$$

2.2. **Faces in $\partial\Omega$.**

Definition 2.6. Let Ω be a properly convex domain. The *face* of $\partial\Omega$ at a point x, which we denote $F_\Omega(x)$, is the set of points $y \in \partial\Omega$ such that x and y lie in an open segment $(a, b) \subset \partial\Omega$.

The *dimension* of a face F is the dimension of a minimal projective subspace containing F; such a minimal subspace is called the *support* of the face and is denoted $\mathrm{supp}(F)$.

A face is always a convex subset of projective space, open in its support. A face is a closed subset of $\partial\Omega$ if and only if it is an extreme point of Ω.

Remark 2.7. Earlier versions of [DGK17] (and of this paper) referred to what we call a *face* as a "stratum." Our current definition of face agrees with the definition used by Islam and Zimmer. Notably, our faces are *not* the same as the *facettes* of Benoist and Benzécri.

In particular, our definition ensures that every face is relatively open in its support, and that every point in the boundary of a properly convex domain Ω is contained in some face.

Definition 2.8. When Λ is a subset of $\partial\Omega$, a *face* of Λ in Ω is a face of $\partial\Omega$ which intersects Λ nontrivially.

We warn the reader that this definition of "face" in Λ depends on both Λ and on the domain Ω whose boundary contains Λ. Often, we will not need to worry about this, due to the following consequence of Lemma 4.1 (1) in [DGK17]:

Lemma 2.9. *Let Ω be a properly convex domain, and let Γ act convex cocompactly on Ω. The full orbital limit set $\Lambda_\Omega(\Gamma)$ is closed and boundary-convex, and contains all of its faces.*

2.3. **The Hilbert metric.** Here we recall the definition of the Hilbert metric, a useful tool for understanding group actions on properly convex domains. See e.g. [Mar14] for more background.

Given four distinct points a, b, c, d in $\mathbb{RP}^1$ (or four points in $\mathbb{RP}^{d-1}$ lying on a single projective line), recall that the *cross-ratio* $[a, b; c, d]$ is given by

$$[a, b; c, d] := \frac{|c-a| \cdot |d-b|}{|b-a| \cdot |d-c|},$$

where the distances are taken in any Euclidean metric on an affine chart containing a, b, c, d.

The cross-ratio is a projective invariant on 4-tuples, and in fact it parameterizes the space of $\mathrm{PGL}(2, \mathbb{R})$-orbits of distinct 4-tuples in $\mathbb{RP}^1$.

Definition 2.10. Let $\Omega \subset \mathbb{P}(V)$ be a properly convex domain. The *Hilbert metric*

$$d_\Omega(\cdot, \cdot) : \Omega^2 \to \mathbb{R}_{\geq 0}$$

is given by the formula

$$d_\Omega(x, y) = \frac{1}{2} \log[a, x; y, b],$$

where a, b are the two points in $\partial\Omega$ such that a, x, y, b lie on a projective line in that order.

When the domain Ω is an ellipsoid of dimension d, the Hilbert metric on Ω recovers the familiar Klein model for hyperbolic space $\mathbb{H}^d$. More generally we have the following:

Proposition 2.11 (See for example [BK553, Section 28])**.** *Let Ω be a properly convex domain. Then:*

1. *The pair (Ω, d_Ω) is a proper metric space.*
2. *If x and y are in Ω, then $[x, y]$ is the image of a geodesic (with respect to d_Ω) joining x and y.*
3. *The group $\mathrm{Aut}(\Omega)$ acts by isometries of d_Ω.*

This implies that $\mathrm{Aut}(\Omega)$ always acts *properly* on Ω. In particular, a subgroup of $\mathrm{Aut}(\Omega)$ is discrete in $\mathrm{PGL}(V)$ if and only if it acts properly discontinuously on Ω.

Part (2) of the above Proposition means that (Ω, d_Ω) is always a geodesic metric space. However, in general it need not be uniquely geodesic—this is one of many ways in which the geometry on a properly convex domain equipped with its Hilbert metric can differ from hyperbolic geometry.

The point of the Hilbert metric is that it allows us to understand many aspects of group actions on convex projective domains in terms of metric geometry; in particular, we may apply the Švarc-Milnor lemma when we have a convex cocompact action on a domain.

The Hilbert metric can also be used to characterize faces in $\partial\Omega$. An easy calculation shows:

Proposition 2.12. *Let Ω be a properly convex domain, let $x \in \partial\Omega$, and fix points $p_1, p_2 \in \Omega$. For any $y \in \partial\Omega$, we have $y \in F_\Omega(x)$ if and only if the Hausdorff distance (with respect to d_Ω) between $[p_1, x)$ and $[p_2, y)$ is finite.*

Since $[p_1, x)$ and $[p_2, y)$ are the images of geodesic rays in (Ω, d_Ω), the above is equivalent to the condition that, if c_x, c_y are unit-speed geodesic rays in (Ω, d_Ω) following projective line segments from p_1, p_2 to x, y, respectively, then

$$d_\Omega(c_x(t), c_y(t)) \leq k$$

for some fixed k independent of $t \in \mathbb{R}_{\geq 0}$.

2.4. **Properly embedded simplices.**

Definition 2.13. A *projective k-simplex in* $\mathbb{RP}^{d-1}$ is the projectivization of the positive linear span of $k+1$ linearly independent vectors in $\mathbb{R}^d$.

A projective k-simplex Δ is an example of a properly convex set in $\mathbb{RP}^{d-1}$. If Δ is the span of standard basis vectors $e_1, \ldots, e_d$, the group $D^+ \subset \mathrm{PGL}(d, \mathbb{R})$ of projectivized diagonal matrices with positive entries (isomorphic to $\mathbb{R}^{d-1}$) acts simply transitively on Δ. Then, any discrete $\mathbb{Z}^{d-1}$ subgroup of D^+ acts properly discontinuously and cocompactly on Δ, so the Švarc-Milnor lemma implies that (Δ, d_Δ) is quasi-isometric to Euclidean space $\mathbb{E}^{d-1}$.

Definition 2.14. Let Ω be a properly convex domain. A convex projective simplex $\Delta \subset \Omega$ is *properly embedded* if $\partial\Delta$ is contained in $\partial\Omega$.

A properly embedded simplex in Ω gives an isometric embedding

$$(\Delta, d_\Delta) \to (\Omega, d_\Omega),$$

which in turn gives a quasi-isometric embedding

$$\mathbb{E}^k \to (\Omega, d_\Omega).$$

Maximal properly embedded simplices in Ω can be thought of as analogues of *maximal flats* in CAT(0) spaces; see e.g. [Ben06a], [IZ21], [IZ19], [Bob20]. However, in general, the metric space (Ω, d_Ω) is not CAT(0); in fact this occurs if and only if Ω is an ellipsoid [KS58].

2.5. **Duality for convex domains.**

Let V be a real vector space. Given a convex set $\Omega \subset \mathbb{P}(V)$, it is often useful to consider the *dual convex set* $\Omega^* \subset \mathbb{P}(V^*)$.

Definition 2.15. Let C be a convex cone in a real vector space V. The *dual convex cone* $C^* \subset V^* - \{0\}$ is

$$C^* = \{\alpha \in V^* : \alpha(x) > 0 \text{ for all } x \in \overline{C} - \{0\}\}.$$

The following is easily verified:

Proposition 2.16. *Let C be a convex cone in a real vector space V.*

(1) C^* is a convex cone in $V^* - \{0\}$.
(2) $C^{**} = C$, under the canonical identification $V^{**} = V$.
(3) C^* is sharp if and only if C has nonempty interior.

If Ω is the projectivization of a convex cone in $\mathbb{P}(V)$, the *dual convex set* is the projectivization Ω^* of $\tilde{\Omega}^*$, where $\tilde{\Omega}$ is any cone over Ω. When Ω is a properly convex domain in $\mathbb{P}(V)$, Ω^* is a properly convex domain in $\mathbb{P}(V^*)$.

In general, Ω^* need not be projectively equivalent to Ω. However, the features of Ω affect the features of Ω^*. For instance, Ω is strictly convex if and only if the boundary of Ω^* is C^1 (and vice versa, since Ω^{**} is naturally identified with Ω). We also note that duality reverses inclusions of convex sets.

If Γ is a subgroup of $\mathrm{PGL}(V)$ preserving Ω, the dual action of Γ on $\mathbb{P}(V^*)$ preserves the dual domain Ω^*. So we can simultaneously view Γ as a subgroup of $\mathrm{Aut}(\Omega)$ and $\mathrm{Aut}(\Omega^*)$.

Remark 2.17. Throughout this paper, we will consistently view the dual projective space as a space of projective hyperplanes in $\mathbb{P}(V)$. That is, we identify an element $[\alpha] \in \mathbb{P}(V^*)$ with the projective hyperplane $\mathbb{P}(\ker \alpha) \subset \mathbb{P}(V)$.

2.6. **Known examples of convex cocompact groups.** Groups with convex cocompact actions in projective space fall into two main classes: the word-hyperbolic groups and the non-word-hyperbolic groups. A consequence of [DGK17] is that a group Γ acting convex cocompactly on some domain Ω is word-hyperbolic if and only if the full orbital limit set of Γ does not contain a nontrivial projective segment. In particular this always holds if Ω itself is strictly convex.

Some of the examples we list below are examples of groups *dividing* domains, meaning that the group Γ acts cocompactly on the entire domain Ω. See [Ben08] for a survey on the topic of convex divisible domains.

2.6.1. *Strictly convex divisible examples.* The simplest example of a strictly convex divisible domain is hyperbolic space $\mathbb{H}^d$. Uniform lattices in $\mathrm{PO}(d,1)$ exist in any dimension $d \geq 1$, so they are examples of hyperbolic groups acting cocompactly on the projective model for $\mathbb{H}^d$ (a round ball in $\mathbb{RP}^d$).

A torsion-free uniform lattice in $\mathrm{PO}(d,1)$ can be viewed as the image of the holonomy representation of a closed hyperbolic d-manifold M. Viewing $\mathrm{PO}(d,1)$ as a subgroup of $\mathrm{PGL}(d+1,\mathbb{R})$ allows us to view the hyperbolic structure on M as a convex projective structure. It is sometimes possible to perturb the subgroup $\Gamma \simeq \pi_1 M$ inside $\mathrm{PGL}(d+1,\mathbb{R})$ to obtain a new discrete group $\Gamma' \simeq \pi_1 M$ in $\mathrm{PGL}(d+1,\mathbb{R})$ which is the holonomy of a different convex projective structure on M. The deformed group Γ' acts cocompactly on some properly convex domain Ω', which in general is not projectively equivalent to a round ball.

Further examples of groups Γ dividing strictly convex domains have been found by Benoist [Ben06b] in dimension 4, using reflection groups, and Kapovich [Kap07] in dimensions $d \geq 4$, by finding convex projective structures on Gromov-Thurston manifolds [GT87]. The Benoist and Kapovich examples share the feature that the dividing group Γ is not isomorphic to any lattice in $\mathrm{PO}(d,1)$—while Γ is word-hyperbolic, the quotient orbifold Ω/Γ carries no hyperbolic structure.

2.6.2. *Non-strictly convex divisible examples.* The simplest examples of non-strictly convex divisible domains are projective k-simplices, which are divided by free abelian groups of rank k. When $k \geq 2$, these simplices are not strictly convex, but we can still decompose the action into strictly convex pieces—the cone over the simplex splits as a sum of strictly convex cones, and each $\mathbb{Z}$ factor acts cocompactly on a summand. So we may wish to find *irreducible* examples. These exist too: uniform lattices in $\mathrm{SL}(d,\mathbb{R})$ act cocompactly on the symmetric space $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$. This symmetric space can be modeled as the projectivization of the set of positive definite symmetric matrices sitting inside the space of $d \times d$ matrices. This set is convex, but not strictly convex whenever $d > 2$.

Other interesting examples of non-strictly convex divisible domains have been discovered. In 2006, Benoist [Ben06a] produced examples of *inhomogeneous* properly convex divisible domains in dimensions $3-7$, divided by non-hyperbolic groups; other examples in dimensions $4-7$ were later found by Choi-Lee-Marquis in [CLM16].

Both of these families of examples essentially come from the theory of reflection groups. Given a Coxeter group Γ acting by reflections in $\mathrm{PGL}(d,\mathbb{R})$, there is a fairly straightforward procedure due to Vinberg [Vin71] which determines whether or not Γ acts cocompactly on some convex domain in projective space. The domain fails to be strictly convex if and only if Γ contains virtually abelian subgroups of rank at least 2 (which happens only when Γ contains a Coxeter subgroup of type $\tilde{A}_n$).

More recently, Blayac-Viaggi [BV23] have constructed additional examples of irreducible non-strictly convex divisible domains in $\mathbb{P}(\mathbb{R}^d)$, for any dimension $d \geq 4$. The construction does not use reflection groups, but rather a combination of arithmetic methods and a procedure known as *projective bending*.

2.6.3. *Non-hyperbolic convex cocompact groups.* In [BDL15], Ballas-Danciger-Lee produce examples of non-hyperbolic groups acting convex cocompactly which do not divide a properly convex domain. These come from deformations of hyperbolic structures on certain cusped hyperbolic 3-manifolds.

None of the groups Γ in the examples in [Ben06a], [CLM16], or [BDL15] are word-hyperbolic, but they are all *relatively* hyperbolic, relative to a family of virtually abelian subgroups of rank at least 2. This situation was studied more generally by Islam-Zimmer in [IZ21], [IZ19]. Islam-Zimmer show that if Γ is hyperbolic relative to virtually abelian subgroups of rank ≥ 2, and Γ acts convex cocompactly on a properly convex domain Ω, the peripheral subgroups of Γ act cocompactly on properly embedded projective simplices in Ω. Moreover, in this situation, the properly embedded maximal simplices in $\mathrm{Cor}_\Omega(\Gamma)$ of dimension at least 2 are *isolated*, and every such maximal simplex has compact quotient by a free abelian subgroup of Γ.

Work of Danciger, Guéritaud, Kassel, Lee, and Marquis [DGK+21] shows that in fact *every* convex cocompact reflection group is either hyperbolic or relatively hyperbolic relative to virtually abelian subgroups. But, this is not true for all non-hyperbolic groups with convex cocompact actions. Uniform lattices in $\mathrm{SL}(d,\mathbb{R})$ provide a counterexample, since the maximal flat subspaces of the Riemannian symmetric space $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$ are *not* isolated (and thus the properly embedded maximal simplices in its projective model are not isolated).

There are also examples of convex cocompact relatively hyperbolic groups which are *not* hyperbolic relative to virtually abelian subgroups. For instance, for every $d \geq 3$, the construction of Blayac-Viaggi mentioned earlier yields groups dividing domains in $\mathrm{PGL}(d,\mathbb{R})$, which are hyperbolic relative to subgroups which are virtually the product of an infinite cyclic group and the fundamental group of a closed hyperbolic $(d-3)$-manifold.

Another construction of relatively hyperbolic convex cocompact groups uses the following (not yet published) result of Danciger-Guéritaud-Kassel:

Proposition 2.18 (See [DGK17, Proposition 12.4] for a statement)**.** *Let $\Gamma_1, \Gamma_2 \subset \mathrm{PGL}(V)$ be groups acting convex cocompactly in $\mathbb{P}(V)$, and suppose that Γ_1, Γ_2 both do not divide any nonempty properly convex open subset in $\mathbb{P}(V)$. Then for some $g \in \mathrm{PGL}(V)$, the group generated by Γ_1 and $g\Gamma_2 g^{-1}$ is isomorphic to the free product $\Gamma_1 * \Gamma_2$, and acts convex cocompactly on $\mathbb{P}(V)$.*

As Danciger-Guéritatud-Kassel observe, it is possible to use this proposition to construct some exotic examples of convex cocompact groups. Here we explain the procedure, assuming Proposition 2.18 holds. Let Γ_1 and Γ_2 be uniform lattices in $\mathrm{SL}(d,\mathbb{R})$. The projective model for the Riemannian symmetric space $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$ is embedded into $\mathbb{P}(V')$, where V' is the vector space of $d \times d$ real matrices. We can in turn embed V' into some vector space V so that $V = V' \oplus V''$ for some complementary subspace V'' with positive dimension. We obtain corresponding embeddings of Γ_1 and Γ_2 into $\mathrm{SL}(V)$ by asking for both of these groups to act via the induced representation $\mathrm{SL}(d,\mathbb{R}) \to \mathrm{SL}(V')$ and trivially on V''.

By [DGK17, Theorem 1.6(E)], Γ_1 and Γ_2 act convex cocompactly in $\mathbb{P}(V)$. Further, since Γ_1 and Γ_2 divide a domain in $\mathbb{P}(V')$, their virtual cohomological dimension is equal to the dimension of $\mathbb{P}(V')$, which prevents either group from dividing any larger-dimensional domain. So these groups satisfy the hypotheses of Proposition 2.18 and we can find a convex cocompact subgroup Γ in $\mathbb{P}(V)$ isomorphic to $\Gamma_1 * \Gamma_2$.

This (abstract) free product is relatively hyperbolic, relative to the collection of conjugates of Γ_1, Γ_2. But, it is not relatively hyperbolic relative to virtually abelian subgroups. One way to see this is that the convex core of Γ in $\mathbb{P}(V)$ must contain the convex core of Γ_1, which is a copy of the projective model for $\mathrm{SL}(d,\mathbb{R})/\mathrm{SO}(d)$; this domain contains many maximal properly embedded simplices which are not isolated.

3. Expansion implies convex cocompactness

The goal of this section is to prove the implication (2) $\implies$ (1) of Theorem 1.5. First let us specify exactly what we mean by "expanding at the faces" of a subset $\Lambda \subset \partial\Omega$.

3.1. Expansion on the Grassmannian.

Recall that a continuous map $f : X \to X$ on a metric space (X, d_X) is said to be C-*expanding* on a subset $U \subset X$, for a constant $C > 1$, if

$$d_X(f(x), f(y)) \geq C \cdot d_X(x, y)$$

for all $x, y \in U$.

Definition 3.1. Let Ω be a properly convex domain in $\mathbb{RP}^{d-1}$, let $\Gamma \subset \mathrm{PGL}(d,\mathbb{R})$ preserve Ω, and let Λ be a Γ-invariant subset of $\partial\Omega$.

Fix a Riemannian metric d_k on each Grassmannian $\mathrm{Gr}(k,d)$. We say that the action of Γ on Ω is *expanding at the faces of* Λ if, for every face F of Λ, there is a constant $C > 1$, an element $\gamma \in \Gamma$, and an open subset $U \subset \mathrm{Gr}(k,d)$ with $\mathrm{supp}(F) \in U$ such that γ is C-expanding on U (with respect to the metric d_k).

If the constant $C > 1$ can be chosen uniformly for all faces F of Λ, then we say the action is C-*expanding at the faces of* Λ or just *uniformly expanding at the faces.*

Remark 3.2. It is conceivable that a group action could be expanding at the faces of Λ with respect to some choice of Riemannian metric d_k on $\mathrm{Gr}(k,d)$, but not with respect to another.

However, if Γ is C-expanding with respect to d_k for a uniform constant C, the choice of metric does not matter: since $\mathrm{Gr}(k,d)$ is compact, all Riemannian metrics on $\mathrm{Gr}(k,d)$ are bilipschitz-equivalent, and when Γ is C-expanding at the faces of Λ, one can apply expanding elements iteratively to see that Γ is also C'-expanding for an arbitrary constant C'.

When the set of supports of $(k-1)$-dimensional faces of Λ is compact in $\mathrm{Gr}(k,d)$ for each k, then a Γ-action is expanding at faces with respect to some choice of metric d_k if and only if it is uniformly expanding at faces with respect to that metric (and hence to every metric). For instance, this is the case when Λ is compact and does not contain any nontrivial segments (so the set of faces is the same as the set of points).

In our context, however, we will not be able to assume this kind of compactness. So, when we discuss expansion, we need to either specify the metric or assume that the expansion is uniform.

Any metric on $\mathbb{RP}^{d-1}$ induces a metric on each $\mathrm{Gr}(k,d)$, by viewing elements of $\mathrm{Gr}(k,d)$ as closed subsets of $\mathbb{RP}^{d-1}$ and taking Hausdorff distance. From this point forward, we will assume that $d_{\mathbb{P}}$ denotes the *angle metric* on projective space, which is induced by a choice of inner product on $\mathbb{R}^d$. Hausdorff distance on $\mathrm{Gr}(k,d)$ (with respect to the angle metric) is a Riemannian metric.

Lemma 3.3. *Let $x \in \mathbb{RP}^{d-1}$, and let $W \in \mathrm{Gr}(k,d)$. There exists $V \in \mathrm{Gr}(k,d)$ so that $x \in V$ and*
$$d_{\mathbb{P}}(x,W) = d_H(V,W),$$
where $d_{\mathbb{P}}$ is the angle metric on projective space, and d_H is the metric induced on $\mathrm{Gr}(k,d)$ by Hausdorff distance.

Proof. If $x \in W$, then we can just take $V = W$, so assume that $d_{\mathbb{P}}(x,W) > 0$. The definition of Hausdorff distance immediately implies that for any V containing x, $d_H(V,W) \geq d_{\mathbb{P}}(x,W)$, so we only need to find some V satisfying the other bound. The diameter of projective space in the angle metric is $\pi/2$, which gives an upper bound on the Hausdorff distance between any two closed subsets of $\mathbb{RP}^{d-1}$. So we only need to consider the case where $d_{\mathbb{P}}(x,W) < \pi/2$.

In this case, we let $W' = x^\perp \cap W$, and then let $V = W' \oplus x$. Let z be the orthogonal projection of x onto W, so that $d_{\mathbb{P}}(x,z) = d_{\mathbb{P}}(x,W)$. Let $\tilde{z}$ and $\tilde{x}$ be unit vector representatives of z and x, respectively, chosen so that if
$$\lambda = \langle \tilde{x}, \tilde{z} \rangle,$$
then
$$d_{\mathbb{P}}(x,z) = \cos^{-1}(\lambda).$$
Let $v \in V - \{0\}$. We want to show that $d_{\mathbb{P}}([v],W) \leq \cos^{-1}(\lambda)$, i.e. that for some $w \in W$,
$$\frac{\langle v, w\rangle}{||v|| \cdot ||w||} \geq \lambda.$$
If $v \in W$, then we can choose $w = v$. Otherwise, we can rescale v in order to write it as $w' + \tilde{x}$, for $w' \in W'$. Then let $w = w' + \tilde{z}$. Note that
$$||w|| = ||v|| = \sqrt{1 + ||w'||^2}.$$
Now we just compute:
$$\begin{aligned} \frac{\langle v, w\rangle}{||v|| \cdot ||w||} = \frac{\langle w' + \tilde{x}, w' + \tilde{z}\rangle}{||v|| \cdot ||w||} &= \frac{\langle \tilde{x}, \tilde{z}\rangle + \langle w', w'\rangle}{1 + ||w'||^2} \\ &\geq \frac{\langle \tilde{x}, \tilde{z}\rangle + \langle \tilde{x}, \tilde{z}\rangle ||w'||^2}{1 + ||w'||^2} = \langle \tilde{x}, \tilde{z}\rangle = \lambda. \end{aligned}$$
$\square$

Most of the work of proving the implication (2) $\implies$ (1) in Theorem 1.5 is contained in the following:

Proposition 3.4. *Let Ω be a convex domain preserved by a group $\Gamma \subset \mathrm{PGL}(d, \mathbb{R})$. Let C be a Γ-invariant subset of Ω, closed in Ω, with ideal boundary $\partial_i C$. Suppose that Γ is expanding at the faces of $\partial_i C$, with respect to the metrics on $\mathrm{Gr}(k, d)$ specified in Lemma 3.3.*

If either

(i) Γ is discrete and Ω is properly convex, or

(ii) Γ is uniformly expanding at the faces of $\partial_i C$,

then Γ acts cocompactly on C.

Proof. Danciger-Guéritaud-Kassel give a proof of this fact in the case where $\partial_i C$ contains no segments (see [DGK17, Lemma 8.7]). Their proof is based on work of Kapovich, Leeb, and Porti [KLP18], which was in turn inspired by Sullivan [Sul85]. Our proof will use essentially the same idea.

We let $d_{\mathbb{P}}$ denote the angle metric on projective space, and we let d_H denote the metric on $\mathrm{Gr}(k, d)$ induced by Hausdorff distance.

For any $\varepsilon > 0$, the set

$$S_\varepsilon = \{x \in C : d_{\mathbb{P}}(x, \partial\Omega) \geq \varepsilon\}$$

is compact. So, supposing for a contradiction that the action of Γ on C is not cocompact, for a sequence $\varepsilon_n \to 0$, there exists x_n so that $\Gamma \cdot x_n$ lies in $C - S_{\varepsilon_n}$.

We start by fixing a constant $E \geq 1$. If Γ is discrete and Ω is properly convex, then we set $E = 1$; otherwise, we let E be less than the uniform expansion constant. In either case, we can replace each x_n with an element in its orbit so that

$$d_{\mathbb{P}}(\gamma x_n, \partial\Omega) \leq E \cdot d_{\mathbb{P}}(x_n, \partial\Omega) \tag{3.1}$$

for all $\gamma \in \Gamma$. This is possible if Γ is discrete and Ω is properly convex because then $\Gamma \cdot x_n$ is a discrete subset of Ω. Otherwise, we choose x_n sufficiently close to a point realizing the maximum distance between $\Gamma \cdot x_n$ and $\partial\Omega$.

Up to a subsequence, x_n converges in $\mathbb{R}\mathrm{P}^{d-1}$ to some $x \in \partial_i C$. Let F be the face of $\partial\Omega$ at x, and let $V \in \mathrm{Gr}(k, d)$ be the support of F.

Let $U \subset \mathrm{Gr}(k, d)$ be an expanding neighborhood of V in $\mathrm{Gr}(k, d)$, with expanding element $\gamma \in \Gamma$ expanding by a constant $E(\gamma) > E$ on U.

Since $\partial\Omega$ is compact and Γ-invariant, there is some $z_n \in \partial\Omega$ so that

$$d_{\mathbb{P}}(\gamma x_n, \gamma z_n) = d_{\mathbb{P}}(\gamma x_n, \partial\Omega).$$

Since $x_n \to x$, and the distance from γx_n to γz_n is at most ε_n, z_n converges to x as well.

Proposition 2.4 implies that there is *some* supporting hyperplane of Ω which intersects z_n. Any such sequence of supporting hyperplanes must sub-converge to a supporting hyperplane of Ω at x. This supporting hyperplane contains V, so there is a sequence $V_n \in \mathrm{Gr}(k, d)$ supporting Ω at z_n, which sub-converges to V.

Since we know γz_n realizes the distance from γx_n to $\partial\Omega$, we must have

$$d_{\mathbb{P}}(\gamma x_n, \partial\Omega) \geq d_{\mathbb{P}}(\gamma x_n, \gamma V_n). \tag{3.2}$$

Then, Lemma 3.3 implies that we can choose subspaces $W_n \in \mathrm{Gr}(k, d)$ containing x_n so that

$$d_{\mathbb{P}}(\gamma x_n, \gamma V_n) = d_H(\gamma W_n, \gamma V_n). \tag{3.3}$$

Since $d_{\mathbb{P}}(\gamma x_n, \gamma V_n)$ converges to 0, $d_H(\gamma W_n, \gamma V_n)$ does as well. Since γ is fixed, and V_n converges to V, W_n also converges to V. So eventually, both V_n and W_n lie in the $E(\gamma)$-expanding neighborhood U of V, meaning that we have

$$d_H(\gamma W_n, \gamma V_n) > E \cdot d_H(W_n, V_n). \tag{3.4}$$

The trivial bound on Hausdorff distance implies that

$$d_H(W_n, V_n) \geq d_{\mathbb{P}}(x_n, V_n). \tag{3.5}$$

Since $x_n \in \Omega$ and $V_n \subset \mathbb{RP}^{d-1} - \Omega$, any $d_{\mathbb{P}}$-geodesic from x_n to V_n must intersect $\partial\Omega$. This implies

$$d_{\mathbb{P}}(x_n, V_n) \geq d_{\mathbb{P}}(x_n, \partial\Omega). \tag{3.6}$$

Putting (3.2), (3.3), (3.4), (3.5), and (3.6) together, we see that

$$d_{\mathbb{P}}(\gamma x_n, \partial\Omega) > E \cdot d_{\mathbb{P}}(x_n, \partial\Omega),$$

which contradicts (3.1) above. □

We need one more lemma before we can show the main result of this section. The statement is closely related to [DGK17, Lemma 6.3], and gives a condition for when a Γ-invariant convex subset of a properly convex domain Ω contains $\mathrm{Cor}_\Omega(\Gamma)$. (The result in [DGK17] is stated for a cocompact action of a group Γ on a convex set C, but the proof only uses Γ-invariance.)

Lemma 3.5. *Let C be a nonempty convex set in Ω whose ideal boundary contains all of its faces, and suppose that $\Gamma \subseteq \mathrm{Aut}(\Omega)$ preserves C. Then $\partial_i C$ contains $\Lambda_\Omega(\Gamma)$, the full orbital limit set of Γ.*

In particular, if Γ is discrete, and the Γ action on C is cocompact, then the action of Γ on Ω is convex cocompact and $\partial_i C = \Lambda_\Omega(\Gamma)$.

Proof. We follow the proof of Lemma 6.3 in [DGK17].

Let $z_\infty \in \Lambda_\Omega(\Gamma)$, which is by definition the limit of a sequence $\gamma_n z$ for some $z \in \Omega$ and a sequence $\gamma_n \in \Gamma$. Fix $y \in C$, and consider the sequence $\gamma_n y$. Since $d(\gamma_n z, \gamma_n y) = d(z, y)$ for all n, Proposition 2.12 implies that up to a subsequence, $\gamma_n z$ and $\gamma_n y$ both converge to points in the same face of $\partial\Omega$. But any accumulation point of $\gamma_n y$ in $\partial\Omega$ lies in $\partial_i C$ and $\partial_i C$ contains its faces, so $z_\infty \in \partial_i C$.

Since $\partial_i C$ contains $\Lambda_\Omega(\Gamma)$, C must contain $\mathrm{Cor}_\Omega(\Gamma)$. [DGK17, Lemma 4.10 (3)] then implies that $\Lambda_\Omega(\Gamma) = \partial_i C$ is closed in C, which means that $\mathrm{Cor}_\Omega(\Gamma)$ is closed in C and the action on $\mathrm{Cor}_\Omega(\Gamma)$ is cocompact. □

Proof of (2) $\implies$ (1) in Theorem 1.5. Let Ω be a properly convex domain, let Γ be a discrete subgroup of $\mathrm{Aut}(\Omega)$, and Λ be a Γ-invariant, closed and boundary-convex subset of $\partial\Omega$ with nonempty convex hull, such that Λ contains all of its faces and Γ is uniformly expanding at the faces of Λ.

Since Λ is boundary-convex and has nonempty convex hull, Λ is exactly the ideal boundary of $\mathrm{Hull}_\Omega(\Lambda)$. So, Proposition 3.4 implies that Γ acts cocompactly on $\mathrm{Hull}_\Omega(\Lambda)$. Since Λ also contains its faces, applying Lemma 3.5 with $C = \mathrm{Hull}_\Omega(\Lambda)$ completes the proof. □

4. Actions on spaces of projective domains

In this section we recall the statement of Benzécri's cocompactness theorem for convex projective domains, as well as prove a version of it (Proposition 4.4) that applies relative to a direct sum decomposition of $\mathbb{R}^d$.

4.1. **The space of projective domains.** Good references for this material include [Gol88] and [Mar14].

Let V be a real vector space. We denote the set of non-empty properly convex open subsets of $\mathbb{P}(V)$ by $\mathcal{C}(V)$. We topologize $\mathcal{C}(V)$ via the metric:

$$d(\Omega_1, \Omega_2) := d_{\mathrm{Haus}}(\overline{\Omega_1}, \overline{\Omega_2}),$$

where $d_{\mathrm{Haus}}(\cdot,\cdot)$ is the Hausdorff distance induced by any metric on $\mathbb{P}(V)$ (the choice of metric on $\mathbb{P}(V)$ does not affect the topology on $\mathcal{C}(V)$).

Definition 4.1. A *pointed properly convex domain* in $\mathbb{P}(V)$ is a pair (Ω, x), where $\Omega \in \mathcal{C}(V)$ and $x \in \Omega$. We denote the set of pointed properly convex domains in $\mathbb{P}(V)$ by $\mathcal{C}_*(V)$, and topologize $\mathcal{C}_*(V)$ by viewing it as a subspace of $\mathcal{C}(V) \times \mathbb{P}(V)$.

$\mathrm{PGL}(V)$ acts on both $\mathcal{C}(V)$ and $\mathcal{C}_*(V)$ by homeomorphisms. We have the following important result:

Theorem 4.2 (Benzécri, [Ben60]). *The action of* $\mathrm{PGL}(V)$ *on* $\mathcal{C}_*(V)$ *is proper and cocompact.*

4.2. **Benzécri relative to a direct sum.** We now let V_a, V_b be subspaces of V so that $V_a \oplus V_b = V$. The decomposition induces natural projection maps $\pi_{V_a} : V \to V_a$ and $\pi_{V_b} : V \to V_b$, as well as a decomposition of the dual V^* into $V_a^* \oplus V_b^*$. Here, and throughout this section, we will identify V_a^*, V_b^* with the linear functionals on V which vanish on V_b, V_a.

When Ω is a convex subset of $\mathbb{P}(V)$ which is disjoint from $\mathbb{P}(V_b)$, we let $\pi_{V_a}(\Omega)$ be the projectivization of $\pi_{V_a}(\tilde{\Omega})$, where $\tilde{\Omega}$ is a cone over Ω. A priori this is only a convex subset of $\mathbb{P}(V_a)$, although we will see (Proposition 4.8) that if Ω is properly convex and open, and $\overline{\Omega}$ is disjoint from $\mathbb{P}(V_b)$, then $\pi_{V_a}(\Omega)$ is properly convex and open in $\mathbb{P}(V_a)$.

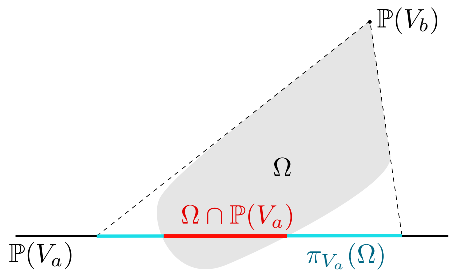

FIGURE 2. The domains $\Omega \cap \mathbb{P}(V_a)$ and $\pi_{V_a}(\Omega)$. In this case, $\pi_{V_a}(\Omega)$ is properly convex even though $\overline{\Omega}$ intersects $\mathbb{P}(V_b)$.

We remark that if $\Omega \cap \mathbb{P}(V_b)$ is nonempty, then $\pi_{V_a}(\Omega)$ is not even well-defined. On the other hand, if $\overline{\Omega} \cap \mathbb{P}(V_b)$ is nonempty, but $\Omega \cap \mathbb{P}(V_b)$ is empty, then $\pi_{V_a}(\Omega)$ does exist, and may or may not be a properly convex subset of $\mathbb{P}(V_a)$.

Definition 4.3. Let $V = V_a \oplus V_b$, and let $\mathcal{K}_a$ be a subset of $\mathcal{C}_*(V_a)$. We define the subset $\mathcal{C}_*(V_a, V_b, \mathcal{K}_a)$ by

$$\mathcal{C}_*(V_a, V_b, \mathcal{K}_a) := \left\{ (\Omega, x) \in \mathcal{C}_*(V) : \begin{array}{l} \mathbb{P}(V_b) \cap \Omega = \emptyset, \\ (\Omega \cap \mathbb{P}(V_a), x) \in \mathcal{K}_a, \\ (\pi_{V_a}(\Omega), x) \in \mathcal{K}_a \end{array} \right\}.$$

The groups $\mathrm{GL}(V_a)$ and $\mathrm{GL}(V_b)$ both have a well-defined action on $\mathcal{C}_*(V)$: we take $g \in \mathrm{GL}(V_a)$ and $h \in \mathrm{GL}(V_b)$ to act by the projectivizations of $g \oplus \mathrm{id}_{V_b}$, $\mathrm{id}_{V_a} \oplus h$ respectively, on $\mathbb{P}(V_a \oplus V_b)$.

Since the $\mathrm{GL}(V_b)$-action on $\mathbb{P}(V)$ fixes $\mathbb{P}(V_a)$ pointwise and commutes with projection to $\mathbb{P}(V_a)$, for any $\mathcal{K}_a \subset \mathcal{C}_*(V_a)$, $\mathrm{GL}(V_b)$ acts on the subset $\mathcal{C}_*(V_a, V_b, \mathcal{K}_a)$. The main result of this section is the following:

Proposition 4.4. *Let V_a, V_b be subspaces of a real vector space V such that $V_a \oplus V_b = V$. For any compact subset $\mathcal{K}_a \subset \mathcal{C}_*(V_a)$, the action of $\mathrm{GL}(V_b)$ on $\mathcal{C}_*(V_a, V_b, \mathcal{K}_a)$ is proper and cocompact.*

4.3. **Convex cones in direct sums.** Before proving Proposition 4.4, we explore some of the properties of convex cones in a vector space V which splits as a direct sum $V = V_a \oplus V_b$.

4.3.1. *Duality.* If $V = V_a \oplus V_b$, a convex cone $C \subset V$ determines two different convex cones in V_a, namely $C \cap V_a$ and $\pi_{V_a}(C)$. The arguments in this section rely heavily on the fact that these projection and intersection operations are in some sense "dual" to each other. Before explaining how this works, we first prove a lemma:

Lemma 4.5. *Let V be a real vector space with $V = V_a \oplus V_b$, and let C be a convex cone intersecting V_b trivially. Then $C^* \cap V_a^* \subseteq \pi_{V_a}(C)^* \cap V_a^*$ and*

$$\overline{\pi_{V_a}(C)^*} \cap V_a^* \subseteq \overline{C^*} \cap V_a^*.$$

Moreover, if $\overline{C} \cap V_b = \{0\}$ then in fact

$$\pi_{V_a}(C)^* \cap V_a^* = C^* \cap V_a^*.$$

Proof. First let $\alpha \in C^* \cap V_a^*$. Let v be any nonzero element of the closure of $\pi_{V_a}(C)$, so that $v + v_2 \in \overline{C}$ for some $v_2 \in V_b$. We know that $\alpha(v + v_2) \neq 0$ and $\alpha(v_2) = 0$, so $\alpha(v) \neq 0$. This shows that α is in $\pi_{V_a}(C)^*$.

Now let $\alpha \in \overline{\pi_{V_a}(C)^*} \cap V_a^* - \{0\}$, and let $v \in C$. We can write $v = v_1 + v_2$ for $v_1 \in V_a$, $v_2 \in V_b$; since we assume C does not intersect V_b, and $C \subset V - \{0\}$ by definition, v_1 is nonzero. Then since $\alpha \in V_a^*$, $\alpha(v) = \alpha(v_1) \neq 0$. So, $\alpha \in \overline{C^*}$.

If we further assume that $\overline{C} \cap V_b = \{0\}$, a similar argument shows that any $\alpha \in \pi_{V_a}(C)^* \cap V_a^*$ is nonzero on any $v \in \overline{C} - \{0\}$, implying $\alpha \in C^*$. ☐

Now, suppose that C_a is a convex cone in $V - \{0\}$, for $V = V_a \oplus V_b$. The intersection $C_a^* \cap V_a^*$ consists of functionals in C_a^* which vanish on V_b. If we know that C_a lies inside of V_a, then any functional on V_a which does not vanish anywhere on $\overline{C_a} - \{0\}$ can be extended by zero on V_b to get an element of $C_a^* \cap V_a^*$. So in this case, $C_a^* \cap V_a^*$ is canonically identified with the dual of the cone C_a *viewed as a cone in V_a*.

We can say this a different way via the following:

Definition 4.6. For each subspace $U \subset V$, we define a "restricted duality" operation D_U, which takes convex cones in U to convex cones in U^* via the dual operation on U. Explicitly, if $C \subset U$ is a convex cone, we let

$$D_U(C) = \{\alpha \in U^* : \alpha(v) > 0 \text{ for all } v \in \overline{C} - \{0\}\}.$$

By definition, we have $D_V(C) = C^*$ for any convex cone $C \subset V$. The reasoning above tells us that when $V = V_a \oplus V_b$, then for any cone $C_a \subset V_a$, we have $D_{V_a}(C_a) = C_a^* \cap V_a^*$.

With this notation, Lemma 4.5 can be restated as:

Lemma 4.7. *Let V be a real vector space with $V = V_a \oplus V_b$, and let C be a convex cone intersecting V_b trivially. Then $D_V(C) \cap V_a^* \subseteq D_{V_a}(\pi_{V_a}(C))$ and $\overline{D_{V_a}(\pi_{V_a}(C))} \subseteq \overline{D_V(C) \cap V_a^*}$. Moreover, if $\overline{C} \cap V_b = \{0\}$, then in fact*

$$D_V(C) \cap V_a^* = D_{V_a}(\pi_{V_a}(C)).$$

As a consequence of this lemma, we note:

Proposition 4.8. *Let C be a sharp (Definition 2.1) open convex cone in a vector space $V = V_a \oplus V_b$. If $\overline{C} - \{0\}$ intersects V_b trivially, then the projection $\pi_{V_a}(C)$ is sharp and open in V_a.*

Proof. Openness is immediate since projection is an open map. Since C is sharp, if $\overline{C}$ does not intersect V_b, then there is some $\alpha \in V^*$ whose kernel contains V_b and does not intersect $\overline{C}$, i.e. $\alpha \in C^* \cap V_a^*$. Since non-intersection with $\overline{C}$ is an open condition, $C^* \cap V_a^*$ is a nonempty open subset of V_a^*. Then Lemma 4.5 implies that $\pi_{V_a}(C)^* \cap V_a^*$ is nonempty and open in V_a^*. So its dual in $V_a^{**} = V_a$ is sharp by part 3 of Proposition 2.16. □

For the rest of the section we will be working with convex domains in $\mathbb{P}(V)$, rather than convex cones in V. The restricted dual operation D_U from Definition 4.6 gives rise to a restricted dual operation on convex domains contained in projective subspaces $\mathbb{P}(U) \subseteq \mathbb{P}(V)$; we also denote this by D_U. Also, recall (from Remark 2.17) that if W is an element in some dual domain $D_V(\Omega) = \Omega^*$, we identify W with a projective hyperplane in $\mathbb{P}(V)$.

4.3.2. *Convex hulls.* If Ω_1, Ω_2 are properly convex subsets of $\mathbb{P}(V)$, we cannot always find a minimal properly convex subset $\Omega \subset \mathbb{P}(V)$ which contains $\Omega_1 \cup \Omega_2$ (that is, convex hulls do not always exist). Here we describe some circumstances under which this is possible.

Definition 4.9. Let Ω_1, Ω_2 be properly convex sets in $\mathbb{P}(V)$. For each $W \in \Omega_1^* \cap \Omega_2^*$, we let $\mathrm{Hull}_W(\Omega_1, \Omega_2)$ denote the convex hull of Ω_1 and Ω_2 in the affine chart $\mathbb{P}(V) - W$.

The set $\mathrm{Hull}_W(\Omega_1, \Omega_2)$ is minimal among all convex subsets of $\mathbb{P}(V) - W$ containing $\Omega_1 \cup \Omega_2$. However, it is possible that for some other $W' \in \Omega_1^* \cap \Omega_2^*$, $\mathrm{Hull}_{W'}(\Omega_1, \Omega_2)$ is not contained in $\mathbb{P}(V) - W$. So, to guarantee minimality among all convex subsets of $\mathbb{P}(V)$, we need a little more:

Lemma 4.10. *If $\Omega_1 \cap \Omega_2$ is nonempty, then for any $W \in \Omega_1^* \cap \Omega_2^*$, $\mathrm{Hull}_W(\Omega_1, \Omega_2)$ is the unique minimal properly convex subset of $\mathbb{P}(V)$ containing $\Omega_1 \cup \Omega_2$.*

Proof. Let A be the affine chart $\mathbb{P}(V) - W$, and let H be any properly convex set containing $\Omega_1 \cup \Omega_2$. Since $\Omega_1 \cap \Omega_2$ is nonempty, $\Omega_1 \cup \Omega_2$ is a connected subset of A, so it is contained in a single connected component C of $H \cap A$. This component is a convex subset of A, so by definition C (hence H) contains $\mathrm{Hull}_W(\Omega_1, \Omega_2)$. □

Lemma 4.10 allows us to define the convex hull of a pair of properly convex sets without reference to a particular affine chart.

Definition 4.11. When Ω_1, Ω_2 are properly convex sets such that $\Omega_1 \cap \Omega_2$ and $\Omega_1^* \cap \Omega_2^*$ are both nonempty, we let $\mathrm{Hull}(\Omega_1, \Omega_2)$ denote the minimal properly convex set containing $\Omega_1 \cup \Omega_2$.

4.4. **Proving Benzécri for direct sums.** We can now begin proving Proposition 4.4. As a first step, we consider the case where $\dim V_a = 1$, i.e. $\mathbb{P}(V_a)$ is identified with a single point in $\mathbb{P}(V)$.

Lemma 4.12. *Let $V = V_b \oplus x$ for a point $x \in \mathbb{P}(V)$. Then $\mathrm{GL}(V_b)$ acts properly and cocompactly on the set of domains*

$$\mathcal{C}_*(x, V_b) := \mathcal{C}_*(x, V_b, \mathcal{C}_*(x)) = \{(\Omega, x) \in \mathcal{C}_*(V) : \mathbb{P}(V_b) \cap \Omega = \emptyset\}.$$

Here $\mathcal{C}_*(x)$ denotes the space of pointed nonempty properly convex domains in $\mathbb{P}(x) \simeq \mathbb{R}\mathrm{P}^0$, so the only nonempty domain in this space is the singleton $\{x\}$.

Statements similar to this lemma can be found in work of Frankel (see [Fra91, Theorem 9.3]) and Benoist (section 2.3 in [Ben03]); Benoist notes that the idea already appears in Benzécri [Ben60].

Proof. Properness follows immediately from the standard Benzécri theorem (Theorem 4.2), since the restriction of a proper action of a group G on X to a closed subgroup H and an H-invariant subset of X is always proper. So, we focus on cocompactness.

Let (Ω_n, x) be a sequence of domains in $\mathcal{C}_*(x, V_b)$. Theorem 4.2 implies that we can find group elements $g_n \in \mathrm{PGL}(V)$ so that the sequence of pointed domains

$$(g_n\Omega_n, g_n x)$$

sub-converges to a pointed domain (Ω, x'). We want to show that these group elements can be chosen to preserve the decomposition $V_b \oplus x$.

We know that V_b lies in $\overline{\Omega_n^*}$, so $g_n V_b$ lies in $g_n\overline{\Omega_n^*}$ for all n, and a subsequence of $g_n V_b$ converges to some $W \in \overline{\Omega^*}$. In particular, W does not contain x'. This means that we can find a sequence of group elements g_n', lying in a fixed compact subset of $\mathrm{PGL}(V)$, so that

$$g_n' \cdot g_n V_b = V_b, \quad g_n' \cdot g_n x = x.$$

Since the g_n' lie in a compact subset of $\mathrm{PGL}(V_b)$, the domains

$$g_n' g_n \Omega_n$$

must also sub-converge to some properly convex domain Ω', which contains x. So we can replace g_n with $g_n' g_n$ to get the desired sequence of group elements. $\square$

Lemma 4.12 gets us partway to proving Proposition 4.4. We see that if Ω is any domain in $\mathcal{C}_*(V_a, V_b, \mathcal{K}_a)$, we can always find some $h \in \mathrm{GL}(V_b)$ so that $h\Omega \cap \mathbb{P}(V_b \oplus x)$ lies in a fixed compact set of domains in $\mathcal{C}(V_b \oplus x)$. This is almost enough to ensure that $h\Omega$ itself lies in a fixed compact set of domains in $\mathcal{C}(V)$. The exact condition we will need is the following:

Lemma 4.13. *Let V be a real vector space, and suppose $V = W_a \oplus V_b \oplus x$, for a point $x \in \mathbb{P}(V)$.*

Let Ω_a, Ω_a' be properly convex domains in $\mathbb{P}(W_a \oplus x)$, and let Ω_b, Ω_b' be properly convex domains in $\mathbb{P}(V_b \oplus x)$, such that

$$x \in \Omega_a \subset \Omega_a',$$
$$x \in \Omega_b \subset \Omega_b'.$$

There exist properly convex domains $\Omega_1 \subset \Omega_2$ in $\mathbb{P}(V)$, with $x \in \Omega_1$, such that any $\Omega \in \mathcal{C}(V)$ disjoint from $\mathbb{P}(W_a)$ and $\mathbb{P}(V_b)$ which satisfies:

(1) $\Omega'_a \supset \pi_{W_a \oplus x}(\Omega)$,
(2) $\Omega_a \subset \Omega \cap \mathbb{P}(W_a \oplus x)$,
(3) $\Omega'_b \supset \pi_{V_b \oplus x}(\Omega)$
(4) $\Omega_b \subset \Omega \cap \mathbb{P}(V_b \oplus x)$,

also satisfies $\Omega_1 \subset \Omega \subset \Omega_2$.

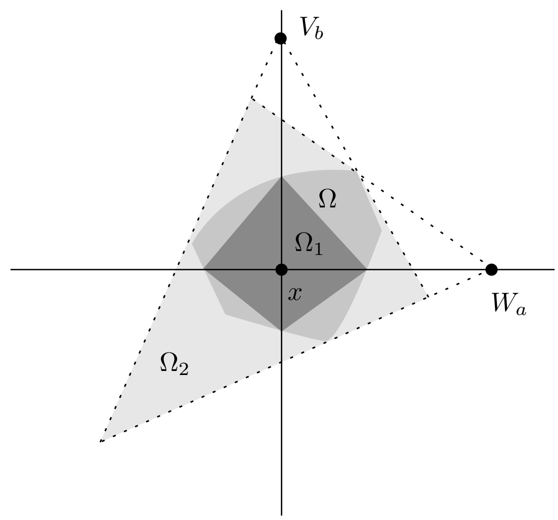

FIGURE 3. Ω fits between a pair of domains Ω_1 and Ω_2, which depend only on the intersections and projections between Ω and $\mathbb{P}(W_a \oplus x)$, $\mathbb{P}(V_b \oplus x)$.

Proof. We know $\Omega_a \cap \Omega_b = \{x\}$. If necessary, we can slightly shrink Ω_a and Ω_b so that $\overline{\Omega_a} \cap \mathbb{P}(W_a) = \emptyset$ and $\overline{\Omega_b} \cap \mathbb{P}(V_b) = \emptyset$, which means that $\mathbb{P}(W_a \oplus V_b)$ can be viewed as an element of $\Omega_a^* \cap \Omega_b^*$. So, the convex hull (Definition 4.11) $\mathrm{Hull}(\Omega_a, \Omega_b)$ of Ω_a, Ω_b exists.

In any affine chart A containing $\Omega_a \cup \Omega_b$, the subspaces $W_a \oplus x$ and $V_b \oplus x$ correspond to pair of transverse affine subspaces intersecting at the point x, which together span all of A. We know Ω_a and Ω_b are open in these subspaces and each contain x, so the interior of their convex hull in A also contains x. Thus, we may define Ω_1 to be the interior of $\mathrm{Hull}(\Omega_a, \Omega_b)$.

To build Ω_2, we consider the "restricted dual" domains

$$D_{W_a \oplus x}(\Omega'_a) \subset \mathbb{P}((W_a \oplus x)^*), \quad D_{V_b \oplus x}(\Omega'_b) \subset \mathbb{P}((V_b \oplus x)^*),$$

which are defined using the restricted dual operation D_U from Definition 4.6. To simplify notation, we write $D(\Omega'_a) = D_{W_a \oplus x}(\Omega'_a)$ and $D(\Omega'_b) = D_{V_b \oplus x}(\Omega'_b)$. Explicitly, we have

$$D(\Omega'_a) = (\Omega'_a)^* \cap \mathbb{P}((W_a \oplus x)^*), \quad D(\Omega'_b) = (\Omega'_b)^* \cap \mathbb{P}((V_b \oplus x)^*).$$

Since Ω'_a and Ω'_b are properly convex subsets of $\mathbb{P}(W_a \oplus x)$ and $\mathbb{P}(V_b \oplus x)$, $D(\Omega'_a)$ and $D(\Omega'_b)$ are open in $\mathbb{P}((W_a \oplus x)^*)$ and $\mathbb{P}((V_b \oplus x)^*)$ (see part 3 of Proposition 2.16).

We also know that x lies in $D(\Omega'_a)^* \cap D(\Omega'_b)^*$. So we can define the convex open set Ω_2^* to be the interior of

$$\mathrm{Hull}_x(D(\Omega'_a), D(\Omega'_b)),$$

using Definition 4.9. Using similar reasoning as for Ω_1, we can see that the interior of this hull is nonempty, because $(W_a \oplus x)^*$ and $(V_b \oplus x)^*$ span V^* and $D(\Omega'_a)$ and $D(\Omega'_b)$ are open subsets of the corresponding projective subspaces.

Let Ω be any domain satisfying the hypotheses of the lemma. Since duality reverses inclusions, we know $D(\Omega'_a) \subseteq D_{W_a \oplus x}(\pi_{W_a \oplus x}(\Omega))$ and $D(\Omega'_b) \subseteq D_{V_b \oplus x}(\pi_{V_b \oplus x}(\Omega))$. Then, Lemma 4.7 implies

$$D(\Omega'_a) \subseteq \overline{D_V(\Omega)} \cap \mathbb{P}((W_a \oplus x)^*),$$
$$D(\Omega'_b) \subseteq \overline{D_V(\Omega)} \cap \mathbb{P}((V_b \oplus x)^*).$$

In particular, $D(\Omega'_a)$ and $D(\Omega'_b)$ are both contained in $\overline{D_V(\Omega)} = \overline{\Omega^*}$. Since $\Omega^{**} = \Omega$ contains x, $\overline{\Omega^*}$ is contained in the affine chart $\mathbb{P}(V^*) - x$. So, $\overline{\Omega^*}$ contains the closure of

$$\mathrm{Hull}_x(D(\Omega'_a), D(\Omega'_b)),$$

meaning Ω^* contains Ω_2^* and Ω is contained in the properly convex set $\Omega_2 = \Omega_2^{**}$. $\square$

Remark 4.14. If $\overline{\Omega}_b$ does not intersect $\mathbb{P}(V_b)$ and $\overline{\Omega}_a$ does not intersect $\mathbb{P}(W_a)$, we can work in the affine chart $\mathbb{P}(V) - \mathbb{P}(W_a \oplus V_b)$, and Lemma 4.13 is equivalent to the fact that if a convex subset C of an affine space has open and bounded projections to and intersections with a pair of complementary affine subspaces, C is itself open and bounded in terms of the size of the projections and intersections.

We do not take this approach because we do *not* want to assume that $\overline{\Omega}_b$ and $\mathbb{P}(V_b)$ are disjoint.

Our next task is to show that we can sometimes replace assumption (3) in Lemma 4.13 with:

(3a) $\Omega'_b \supset \Omega \cap \mathbb{P}(V_b \oplus x)$.

This will be done in Proposition 4.16 below. We start with some Euclidean geometry.

We endow $\mathbb{R}^d$ with its standard inner product. For a subspace $W \subseteq \mathbb{R}^d$, we let $\pi_W : \mathbb{R}^d \to W$ denote the orthogonal projection, and for $R > 0$, let $B(R)$ denote the open ball around the origin of radius R.

Lemma 4.15. *Let Ω be a convex subset of $\mathbb{R}^d$ containing the origin, and let W be a subspace of $\mathbb{R}^d$.*

Suppose that there are $R_1, R_2 > 0$ so that:

- $B(R_1) \cap W^\perp \subset \Omega \cap W^\perp$,
- $\pi_{W^\perp}(\Omega) \subset B(R_2)$.

Then there exists a linear map $f : \mathbb{R}^d \to \mathbb{R}^d$, depending only on R_1 and R_2, so that $\pi_W(\Omega) \subset f(\Omega \cap W)$.

Proof. Let p be any point in $\pi_W(\Omega)$, and let z be some point in Ω so that $\pi_W(z) = p$. We can write $z = p + y$ for $y \in \pi_{W^\perp}(\Omega)$.

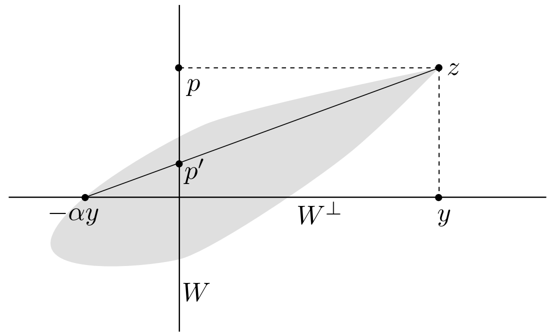

FIGURE 4. Illustration for the proof of Lemma 4.15. The ratio $||p||/||p'||$ is bounded in terms of α.

Let ℓ be the line through the origin passing through y. For some $\alpha > 0$, we know that ℓ intersects $\overline{(\Omega \cap W^\perp)} - B(R_1)$ at $y' = -\alpha y$. Note that

$$\alpha = \frac{||y'||}{||y||} > \frac{R_1}{R_2}.$$

Since Ω is convex and contains $\Omega \cap W^\perp$, it contains the open line segment

$$\{t(-\alpha y) + (1-t)(y+p) : t \in (0,1)\}.$$

This line segment passes through W when $t = \frac{1}{1+\alpha}$, meaning that Ω must contain the point

$$p' = \left(1 - \frac{1}{1+\alpha}\right) p.$$

Since Ω contains the origin, it also contains

$$\left(1 - \frac{1}{1 + R_1/R_2}\right) p = \frac{R_1}{R_1 + R_2} p.$$

This point lies in $\Omega \cap W$, meaning that p lies in $R_3 \cdot (\Omega \cap W)$ where

$$R_3 := \frac{R_1 + R_2}{R_1}.$$

So we can take our map f to be the linear rescaling about the origin by R_3. $\square$

Proposition 4.16. *Let $V = W_a \oplus V_b \oplus x$, for $x \in \mathbb{P}(V)$.*

Let Ω_a, Ω'_a be properly convex domains in $\mathbb{P}(W_a \oplus x)$, and let Ω''_b be a properly convex domain in $\mathbb{P}(V_b \oplus x)$ such that

$$x \in \Omega_a \subset \Omega'_a, \quad x \in \Omega''_b.$$

If $\overline{\Omega'_a}$ does not intersect $\mathbb{P}(W_a)$, then there exists a properly convex domain Ω'_b in $\mathbb{P}(V_b \oplus x)$ so that any $\Omega \in \mathcal{C}(V)$ which satisfies $\Omega \cap \mathbb{P}(V_b) = \emptyset$ and

(1) $\Omega'_a \supset \pi_{W_a \oplus x}(\Omega)$,
(2) $\Omega_a \subset \Omega \cap \mathbb{P}(W_a \oplus x)$,
(3a) $\Omega''_b \supset \Omega \cap \mathbb{P}(V_b \oplus x)$

also satisfies

(3) $\Omega'_b \supset \pi_{V_b \oplus x}(\Omega)$.

Proof. Let $H = W_a \oplus V_b$, and consider the affine chart $A = \mathbb{P}(V) - \mathbb{P}(H)$. We can choose coordinates and a Euclidean metric on this affine chart so that $W_a \oplus x$ and $V_b \oplus x$ map to complementary orthogonal subspaces W_a, V_b of A, meeting at the origin. In these coordinates, the projectivizations of the projection maps $\pi_{W_a \oplus x}$, $\pi_{V_b \oplus x}$ correspond to the orthogonal projections to W_a and V_b, respectively.

Since $\mathbb{P}(W_a)$ does not intersect $\overline{\Omega'_a}$, the images of Ω'_a and Ω_a in A are both bounded open convex subsets of W_a.

Let Ω be a properly convex domain not intersecting $\mathbb{P}(V_b)$ and satisfying assumptions (1), (2), (3a). Since $\pi_{W_a \oplus x}(\Omega)$ is contained in A, Ω cannot intersect $\mathbb{P}(W_a \oplus V_b)$, so Ω is contained in the affine chart A (although its closure need not be).

In particular, $\Omega \cap \mathbb{P}(V_b \oplus x)$ is contained in the unique connected component of $\Omega''_b \cap A$ which contains x. So, by replacing Ω''_b with this connected component, we may assume that the image of Ω''_b in A is a convex open subset of V_b.

Lemma 4.15 then implies that there is an affine map $f : A \to A$, depending only on Ω_a and Ω'_a, so that

$$\pi_{V_b \oplus x}(\Omega) \subseteq f(\Omega''_b).$$

So, we can take Ω'_b to be the properly convex domain $f(\Omega''_b)$. $\square$

We are now ready to prove Proposition 4.4.

Proof of Proposition 4.4. As in the proof of Lemma 4.12, properness is immediate from the Benzécri cocompactness theorem, so we just need to show cocompactness. We let V_a, V_b, and $\mathcal{K}_a \subset \mathcal{C}_*(V_a)$ be as in the statement of the theorem. Let (Ω_n, x_n) be a sequence of properly convex domains in $\mathcal{C}_*(V_a, V_b, \mathcal{K}_a)$. We can choose a subsequence so that $x_n \to x$. Our goal is to find a pair of properly convex domains Ω_1, Ω_2 (with $x \in \Omega_1$) and $h_n \in \mathrm{GL}(V_b)$, so that up to a subsequence,

$$\Omega_1 \subset h_n \cdot \Omega_n \subset \Omega_2.$$

This will be sufficient, because $x_n \in \mathbb{P}(V_a)$, so $h_n x_n = x_n$ converges to x and $h_n \Omega_n$ sub-converges to some properly convex domain Ω containing $\Omega_1 \ni x$.

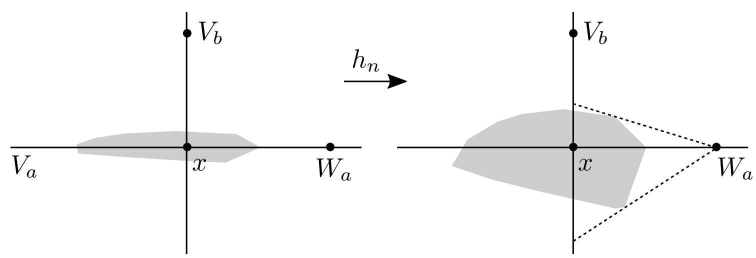

FIGURE 5. Applying an element $h_n \in \mathrm{GL}(V_b)$ "rescales" in $\mathbb{P}(V_b \oplus x)$ about x; if the size of the intersection $\Omega \cap \mathbb{P}(V_b \oplus x)$ is bounded, then the size of the projection to $V_b \oplus x$ (with respect to the decomposition $V = W_a \oplus V_b \oplus x$) is also bounded (Proposition 4.16).

Consider the sequence of domains $\Omega'_n = \Omega_n \cap \mathbb{P}(V_b \oplus x)$. We know $\mathbb{P}(V_b)$ is disjoint from Ω_n for all n. So, Lemma 4.12 implies that we can find $h_n \in \mathrm{GL}(V_b)$ so that the domains $h_n \Omega'_n$ sub-converge in $\mathcal{C}(V_b \oplus x)$ to some domain Ω' in $\mathbb{P}(V_b \oplus x)$.

In particular, up to a subsequence, we can find fixed domains $\Omega_b, \Omega_b'' \subset \mathbb{P}(V_b \oplus x)$ such that for all n,

$$x \in \Omega_b \subset h_n\Omega_n' \subset \Omega_b''.$$

Since the intersections $\Omega_n \cap \mathbb{P}(V_a)$ and projections $\pi_{V_a}(\Omega_n)$ both lie in a fixed compact set in $\mathcal{C}(V_a)$, we can also assume that there are domains $\Omega_a, \Omega_a' \in \mathcal{C}(V_a)$ so that for all n,

$$\Omega_a \subset \Omega_n \cap \mathbb{P}(V_a), \quad \Omega_a' \supset \pi_{V_a}(\Omega_n).$$

Since the action of any $h_n \in \mathrm{GL}(V_b)$ fixes V_a pointwise and commutes with projection to V_a, this immediately implies that for all n,

$$\Omega_a \subset h_n\Omega_n \cap \mathbb{P}(V_a), \quad \Omega_a' \supset \pi_{V_a}(h_n\Omega_n).$$

Fix a subspace $W_a \subset V_a$ so that $V_a = W_a \oplus x$ and $\mathbb{P}(W_a)$ does not intersect the closure of Ω_a'. This allows us to define a projection map $\pi_{V_b \oplus x} : V \to V_b \oplus x$, whose kernel is W_a. Proposition 4.8 implies that $\pi_{V_b\oplus x}(h_n\Omega_n)$ is a properly convex open subset of $\mathbb{P}(V_b \oplus x)$, and Proposition 4.16 implies that for all n, $\pi_{V_b\oplus x}(h_n\Omega_n)$ is contained in a properly convex domain $\Omega_b' \subset \mathbb{P}(V_b \oplus x)$, depending only on Ω_a, Ω_a', and Ω_b''. Then we can apply Lemma 4.13 to the domains $\Omega_a, \Omega_a', \Omega_b$, Ω_b' to finish the proof. $\square$

5. Cocompactness implies expansion

The main goal of this section is to prove the implication (1) $\implies$ (2) of Theorem 1.5. In fact we will prove a slightly more general statement:

Proposition 5.1. *Let C be a convex subset of a properly convex domain Ω, and suppose that $\Gamma \subseteq \mathrm{Aut}(\Omega)$ acts cocompactly on C. Then Γ is uniformly expanding at the faces of the ideal boundary of C.*

Afterwards, we will use some of the ideas arising in the proof to show that a version of "north-south dynamics" holds for certain sequences of elements in a convex cocompact group (Proposition 5.15).

5.1. Pseudo-loxodromic elements.

Our main inspiration comes from an observation in Sullivan's study [Sul79] of conformal densities on $\mathbb{H}^d$: if $x_0 \in \mathbb{H}^d$ is a basepoint defining a visual metric on $\partial\mathbb{H}^d$, and γ is any isometry of $\mathbb{H}^d$ not fixing x_0, then γ expands a small ball in $\partial\mathbb{H}^d$ at the endpoint of the geodesic ray from x_0 to $\gamma^{-1}x_0$, with expansion constant related to $d(x_0, \gamma^{-1}x_0)$.

This observation relies on the fact that, given distinct points $x, y \in \mathbb{H}^d$, there is a loxodromic isometry taking x to y whose axis is the geodesic joining x and y. The exact analogue of this fact for properly convex domains does not hold in general, since there is no reason to expect even the full automorphism group of a properly convex domain to act transitively on the domain. However, instead of looking for actual automorphisms of the domain, we can instead look for elements of $\mathrm{PGL}(d, \mathbb{R})$ that do not perturb the domain "too much." We make this precise below.

Definition 5.2. Let $\Omega \subset \mathbb{R}\mathrm{P}^{d-1}$ be a properly convex domain, and let $\mathcal{K}$ be a compact subset of $\mathcal{C}(\mathbb{R}^d)$ containing Ω. An element $g \in \mathrm{PGL}(d, \mathbb{R})$ is a $\mathcal{K}$-*pseudo-automorphism* of Ω if $g\Omega \in \mathcal{K}$.

Definition 5.3. Let $\Omega \subset \mathbb{RP}^{d-1}$ be a properly convex domain. For a compact subset $\mathcal{K} \subset \mathcal{C}(\mathbb{R}^d)$ containing Ω, we say that a $\mathcal{K}$-pseudo-automorphism $g \in \mathrm{PGL}(d, \mathbb{R})$ is *$\mathcal{K}$-pseudo-loxodromic* if there is a g-invariant direct sum decomposition

$$\mathbb{R}^d = V_- \oplus V_0 \oplus V_+,$$

where:

(i) the subspaces V_-, V_+ are positive eigenspaces of g and supporting subspaces of Ω,
(ii) the convex hull of $\mathbb{P}(V_+) \cap \partial\Omega$ and $\mathbb{P}(V_-) \cap \partial\Omega$ has nonempty intersection with Ω, and
(iii) the projective subspace $\mathbb{P}(V_- \oplus V_+)$ intersects every Ω' in $\mathcal{K}$.

The subspaces V_- and V_+ are referred to as *endpoints* of g. The projective subspace $\mathbb{P}(V_- \oplus V_+)$ is the *axis* of the pseudo-loxodromic, and V_0 is the *neutral subspace*.

A pseudo-loxodromic element preserves its axis $\mathbb{P}(V_- \oplus V_+)$. When V_- and V_+ are points in $\mathbb{RP}^{d-1}$, this axis is an actual projective line.

We do *not* assume that an individual pseudo-loxodromic element attracts points on its axis towards either of its endpoints, since we are only interested in the dynamics of *sequences* of pseudo-loxodromics.

If g_n is a sequence of $\mathcal{K}$-pseudo-loxodromic elements with common endpoints, then, up to a subsequence, the domains $g_n\Omega$ converge to a domain Ω_∞ in $\mathcal{K}$ which intersects the common axis. In fact, we observe:

Proposition 5.4. *Let g_n be a sequence of $\mathcal{K}$-pseudo-loxodromic elements with common endpoints V_+, V_-. If $g_n\Omega$ converges to Ω_∞, then Ω_∞ contains the relative interior of the convex hull (in Ω) of $\mathbb{P}(V_+) \cap \partial\Omega$ and $\mathbb{P}(V_-) \cap \partial\Omega$.*

Proof. Let W denote the subspace $V_+ \oplus V_-$. We know that the intersection $\mathbb{P}(W) \cap \overline{\Omega_\infty}$ is either contained in a face of Ω_∞, or else its relative interior is contained in Ω_∞. It must be the latter, since we know Ω_∞ is in $\mathcal{K}$ and by definition $\Omega' \cap \mathbb{P}(W)$ is nonempty for every $\Omega' \in \mathcal{K}$.

We let C denote the convex hull of $\mathbb{P}(V_+) \cap \partial\Omega$ and $\mathbb{P}(V_-) \cap \partial\Omega$. It now suffices to show that the relative interior of C is contained in the relative interior of $\mathbb{P}(W) \cap \Omega_\infty$. First, suppose that every subsequence of the restriction of g_n to W has a further subsequence which converges to some $g \in \mathrm{PGL}(W)$. In this case, we know $g\overline{\Omega} \cap \mathbb{P}(W) = \overline{\Omega_\infty} \cap \mathbb{P}(W)$; then gC lies in the relative interior of Ω_∞ because C lies in the relative interior of $\Omega \cap \mathbb{P}(W)$ by assumption.

Otherwise, the ratio of the eigenvalues of g_n on V_+ and V_- is unbounded. We let $W' \subset W$ be the vector space whose projectivization $\mathbb{P}(W')$ is the projective span of $\Omega \cap \mathbb{P}(V_+)$ and $\Omega \cap \mathbb{P}(V_-)$. The unboundedness of the eigenvalue ratio of g_n implies that $g_n\overline{\Omega} \cap \mathbb{P}(W)$ must converge to a subset of $\mathbb{P}(W')$. But this limit is $\overline{\Omega_\infty} \cap \mathbb{P}(W)$, which has nonempty relative interior in $\mathbb{P}(W)$ because $\mathbb{P}(W) \cap \Omega_\infty$ is nonempty. This is only possible if $W' = W$, which means that C also has nonempty relative interior in $\mathbb{P}(W)$. Since $\overline{C} \subset \overline{\Omega_\infty} \cap \mathbb{P}(W)$ we must therefore have $C \subset \Omega_\infty \cap \mathbb{P}(W)$. ☐

Definition 5.5. Let Ω be a properly convex domain, and let g_n be a sequence of $\mathcal{K}$-pseudo-loxodromic elements with common endpoints V_+, V_- and common neutral subspace V_0. We say that V_- is a *repelling endpoint* of the sequence g_n if there is a sequence

$$x_n \in \Omega \cap \mathbb{P}(V_- \oplus V_+)$$

such that $g_n x_n = x$ for some $x \in \Omega$, and $x_n \to x_- \in \partial\Omega$ with

$$V_- = \operatorname{supp} F_\Omega(x_-).$$

5.2. **Existence of repelling pseudo-loxodromics.** We will use pseudo-loxodromics to state an analogue (Lemma 5.7) of the fact that any two points in $\mathbb{H}^d$ can be joined by the axis of a loxodromic isometry. First, we need a lemma:

Lemma 5.6. *Let x_+, x_- be a pair of points in the boundary of a properly convex domain $\Omega \subset \mathbb{P}(V)$ such that $(x_-, x_+) \subseteq \Omega$. Let $\mathbb{P}(H_+)$, $\mathbb{P}(H_-)$ be supporting hyperplanes of Ω at x_+, x_-. Let $V_- = \operatorname{supp}(F_\Omega(x_-))$, and let $W = V_- \oplus x_+$.*

There exists a (possibly trivial) subspace $H_0 \subset H_+ \cap H_-$ such that

(1) $H_- = H_0 \oplus V_-$, and
(2) $\pi_W(\Omega)$ is properly convex, where $\pi_W : V \to W$ is the projection with kernel H_0.

Note that while $\mathbb{P}(H_0)$ does not intersect Ω, the intersection $\mathbb{P}(H_0) \cap \overline{\Omega}$ may be nonempty.

Proof. First suppose that $V_- = x_-$. In this case, we take $H_0 = H_+ \cap H_-$, and $\pi_W(\Omega)$ is exactly the line segment (x_-, x_+). On the other hand, if V_- has codimension one, then $W = V_- \oplus x_+ = V$ and we can take H_0 to be trivial, so π_W is the identity map.

So now suppose that V_- is neither a single point nor a hyperplane in V. Consider the properly convex set $\Omega_- = \partial\Omega \cap H_-$. We know $\mathbb{P}(H_+ \cap H_-)$ is a codimension-one projective subspace of $\mathbb{P}(H_-)$. Because $(x_-, x_+) \subseteq \Omega$, $H_+ \cap H_-$ does not contain V_-.

We also know $\mathbb{P}(H_+ \cap H_-)$ intersects $\overline{\Omega_-}$ in a (possibly empty) properly convex set. We know the projective subspace V_- has dimension $k \geq 1$ and positive codimension in H_-, so there exists a codimension-k projective subspace of $H_+ \cap H_-$ which does not intersect $\overline{\Omega_-}$ or V_-. Let H_0 be such a subspace; since H_0 is disjoint from $\overline{\Omega}$, we are done by Proposition 4.8. ☐

The following lemma is the main technical result in this section. It implies in particular that every face in the boundary of a properly convex domain is the repelling endpoint of *some* sequence of $\mathcal{K}$-pseudo-loxodromics.

Lemma 5.7. *Let Ω be a properly convex domain, let $x_- \in \partial\Omega$, and let L be a projective line intersecting Ω, joining x_- with some $x_+ \in \partial\Omega$, $x_+ \neq x_-$. Let $F_- = F_\Omega(x_-)$.*

For any sequence $\{x_n\} \subset L$, with $x_n \to x_-$, up to a subsequence, there exists a compact set $\mathcal{K} \subset \mathcal{C}(\mathbb{R}^d)$, a subspace $H_0 \subset \mathbb{R}^d$, and a sequence of $\mathcal{K}$-pseudo-loxodromic elements g_n in $\mathrm{PGL}(d, \mathbb{R})$, with endpoints $\operatorname{supp}(F_-)$ and x_+ and neutral subspace H_0, such that $g_n x_n = x$ for a fixed $x \in L \cap \Omega$.

Proof. Our strategy is to start with the case that F_- is codimension-one (so the neutral subspace H_0 is trivial), and then use Proposition 4.4 to extend to the general case.

F_- is codimension-one. Let V_- be the support of F_-. For each n, we let $s_n \in \mathrm{GL}(d, \mathbb{R})$ be the diagonal map

$$\lambda_n \mathrm{id}_{x_+} \oplus \mathrm{id}_{V_-} = \begin{bmatrix} \lambda_n & \\ & \mathrm{id}_{V_-} \end{bmatrix}$$

acting on $x_+ \oplus V_-$, where $\lambda_n \to \infty$ is chosen so that $s_n x_n = x$ for a fixed $x \in L \cap \Omega$.

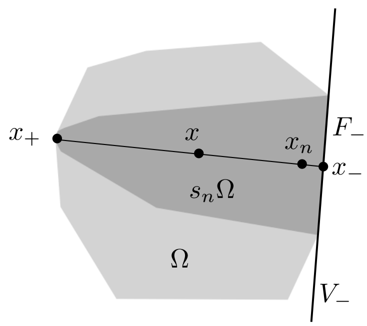

FIGURE 6. Since s_n attracts towards x_+ and repels from V_-, $s_n\Omega$ converges to the convex hull of F_- and x_+.

The sequence of domains $s_n \cdot \Omega_n$ converges to a cone over F_-, with a cone point at x_+ (see Figure 6). Since F_- is a codimension-one face of Ω, this cone is a properly convex domain containing x in its interior.

The general case. Let V_- be the support of F_-, and let H_+, H_- be supporting hyperplanes of Ω at x_+, F_-. Let $W = V_- \oplus x_+$. We choose a subspace $H_0 \subset H_+ \cap H_-$ as in Lemma 5.6 so that $H_- = V_- \oplus H_0$ and $\pi_W(\Omega)$ is properly convex, where $\pi_W : V \to W$ is the projection with kernel H_0.

The domains

$$\Omega \cap \mathbb{P}(W), \quad \pi_W(\Omega)$$

are both properly convex open subsets of $\mathbb{P}(W)$ containing F_- as a codimension-one face in their boundaries. Using the argument from the previous case, we can find group elements $s_n \in \mathrm{GL}(W)$ so that

$$s_n \cdot (\Omega \cap P(W)), \quad s_n \cdot \pi_W(\Omega)$$

both converge to properly convex domains in $\mathbb{P}(W)$ containing a fixed $x = s_n x_n$ in Ω.

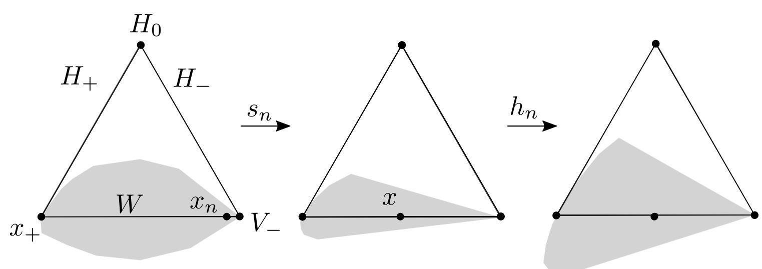

FIGURE 7. To build the sequence of pseudo-loxodromic elements g_n, we push x_n away from x_- with $s_n \in \mathrm{GL}(W)$, ensuring that $s_n\Omega \cap \mathbb{P}(W)$ and $\pi_W(s_n\Omega)$ converge, and then use a "correcting" element $h_n \in \mathrm{GL}(H_0)$ to keep the domain from degenerating. Both s_n and h_n preserve the decomposition $\mathbb{R}^d = x_+ \oplus H_0 \oplus V_-$.

We extend s_n linearly to the map $s_n \oplus \mathrm{id}_{H_0}$ on $W \oplus H_0$. Consider the sequence of properly convex domains

$$\Omega_n = (s_n \oplus \mathrm{id}_{H_0}) \cdot \Omega.$$

Since $s_n \oplus \mathrm{id}_{H_0}$ commutes with projection to W and intersection with W, the sequences of pointed properly convex domains

$$(\Omega_n \cap \mathbb{P}(W), x), \quad (\pi_W(\Omega_n), x)$$

both converge in $\mathcal{C}_*(W)$. In particular, both of these sequences are contained in a fixed compact $\mathcal{K}_W \subset \mathcal{C}_*(W)$, and the pointed domains (Ω_n, x) all lie in the subset

$$\mathcal{C}_*(W, H_0, \mathcal{K}_W)$$

from Definition 4.3.

Then, Proposition 4.4 (applied to the decomposition $\mathbb{R}^d = W \oplus H_0$) tells us that there is a sequence of group elements $h_n \in \mathrm{GL}(H_0)$ such that the pointed properly convex domains

$$(\mathrm{id}_W \oplus h_n) \cdot (\Omega_n, x)$$

lie in a fixed compact $\mathcal{K}$ in $\mathcal{C}_*(\mathbb{R}^d)$.

Then, we can take our sequence of $\mathcal{K}$-pseudo-loxodromic elements g_n to be the projectivizations of $(\mathrm{id}_W \oplus h_n) \cdot (s_n \oplus \mathrm{id}_{H_0}) = (s_n \oplus h_n)$. □

Next we examine some of the dynamical behavior of pseudo-loxodromic sequences that have a repelling endpoint. Let V be a normed vector space. For any $g \in \mathrm{GL}(V)$, recall that the *norm* and *conorm* of g on V are defined by

$$||g|| = \sup_{v \in V - \{0\}} \frac{||gv||}{||v||}, \quad \mathbf{m}(g) = \inf_{v \in V - \{0\}} \frac{||gv||}{||v||}.$$

Proposition 5.8. *Let g_n be a sequence of $\mathcal{K}$-pseudo-loxodromic elements with common endpoints V_+, V_- and common neutral subspace V_0, and suppose that V_- is a repelling endpoint (Definition 5.5) of the sequence g_n. Let $U_+ = V_+ \oplus V_0$. The sequence g_n satisfies*

$$\frac{\mathbf{m}(g_n|_{U_+})}{||g_n|_{V_-}||} \to \infty. \tag{5.1}$$

The ratio in (5.1) can be computed by fixing a norm on $\mathbb{R}^d$, and then choosing a lift of each g_n in $\mathrm{GL}(d, \mathbb{R})$. The value of this ratio does not depend on the choice of lift, and the asymptotic behavior of the ratio does not depend on the choice of norm.

Proof. We can fix lifts $\tilde{g}_n$ of g_n in $\mathrm{GL}(d, \mathbb{R})$ which restrict to the identity on V_-. Our goal is then to show that

$$\mathbf{m}(\tilde{g}_n|_{U_+}) \to \infty,$$

or equivalently, that

$$||\tilde{g}_n^{-1}|_{U_+}|| \to 0.$$

Suppose otherwise, so that for a sequence $v_n \in U_+$ with $||v_n|| = 1$, there is some $\varepsilon > 0$ so that

$$||\tilde{g}_n^{-1} \cdot v_n|| \geq \varepsilon.$$

Let $x_n \in \Omega \cap \mathbb{P}(V_+ \oplus V_-)$ be a sequence so that $g_n x_n = x$ for some $x \in \Omega$ and $x_n \to x_-$, where V_- is the support of $F_\Omega(x_-)$. We can choose a subsequence so that

$g_n\Omega$ converges to some properly convex domain Ω_∞. We know Ω_∞ contains x by Proposition 5.4, so let U be an open neighborhood of x whose closure is contained in Ω_∞. We can find a lift $\tilde{x}$ of x in $\mathbb{R}^d$ so that the projectivizations of each vector

$$\tilde{x} \pm v_n$$

lie in U, and thus in Ω_n for all sufficiently large n. Since $\tilde{g}_n$ restricts to the identity on V_-, the sequence $\tilde{g}_n^{-1}\tilde{x}$ converges to a lift $\tilde{x}_-$ of x_-.

Then, up to a subsequence, the sequence of pairs of vectors

$$\tilde{g}_n^{-1} \cdot (\tilde{x} \pm v_n)$$

lies in a lift $\tilde{\Omega}$ of Ω, and converges in $\mathbb{R}^d$ to $\tilde{x}_- \pm v_\infty$, where $v_\infty \in U_+$ has norm at least ε. This pair of points spans a nontrivial projective line segment in $\overline{\Omega}$ whose interior intersects the face $F_\Omega(x_-)$ only at x_-, contradicting the definition of $F_\Omega(x_-)$. □

Proposition 5.8 implies in particular that a sequence of $\mathcal{K}$-pseudo-loxodromic elements with repelling subspace V_- attracts generic points in $\mathbb{RP}^{d-1}$ to the projective subspace $\mathbb{P}(U_+)$, and repels generic points away from $\mathbb{P}(V_-)$. In fact, because the subspaces U_+ and V_- are transverse, and g_n preserves each of them, the proposition also implies that the sequence g_n has *expansion* behavior on the Grassmannian $\mathrm{Gr}(k, d)$ in a neighborhood of V_-. To see this, we need an estimate relating the ratio appearing in (5.1) to the metric behavior of g_n on the Grassmannian.

To state the estimate, we choose an inner product on $\mathbb{R}^d$, and endow $\mathbb{RP}^{d-1}$ with the metric d_s obtained by setting $d_s(x, y)$ to be the sine of the minimum angle between lifts of x and y in $\mathbb{R}^d$. Then we let d_k denote the metric on $\mathrm{Gr}(k, d)$ induced by Hausdorff distance with respect to d_s.

Lemma 5.9 (See the appendix in [BPS19], specifically Lemma A.10)**.** *Let $U_+ \in \mathrm{Gr}(d-k, d)$ and $U_- \in \mathrm{Gr}(k, d)$ be transverse subspaces of $\mathbb{R}^d$, and let $g \in \mathrm{PGL}(d, \mathbb{R})$. Suppose that for some $\alpha > 0$, we have $\angle(U_+, U_-) > \alpha$ and $\angle(gU_+, gU_-) > \alpha$.*

Then, for constants $b > 0$ and $\delta > 0$ (depending only on α), if $W_1, W_2 \in \mathrm{Gr}(k, d)$ satisfy $d_k(W_i, V_-) < \delta$, then:

$$d_k(gW_1, gW_2) \geq b\frac{\mathbf{m}(g|_{U_+})}{||g|_{U_-}||}d(W_1, W_2).$$

We can use this lemma to estimate the expansion behavior of elements in our sequence g_n:

Corollary 5.10. *Let g_n be a sequence of $\mathcal{K}$-pseudo-loxodromic elements with common endpoints V_+, V_- and common neutral subspace V_0, and suppose that V_- is a repelling endpoint of the sequence, lying in $\mathrm{Gr}(k, d)$.*

Then for any Riemannian metric on $\mathrm{Gr}(k, d)$, and any $E > 1$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, g_n is E-expanding on some neighborhood of V_- in $\mathrm{Gr}(k, d)$.

Proof. This follows directly from Proposition 5.8 and Lemma 5.9, taking $U_+ = V_+ \oplus V_0$ and $U_- = V_-$ and exploiting the fact that each g_n preserves the decomposition $\mathbb{R}^d = V_- \oplus U_+$. □

Remark 5.11. In order to apply Lemma 5.9 to our situation, we need to know that our sequence of group elements g_n actually preserves the decomposition $V_- \oplus U_+$ (or at least that the sequence of decompositions $g_nV_- \oplus g_nU_+$ does not degenerate).

This is why it is useful to have the additional control afforded by the pseudo-loxodromic sequences we have constructed—it is not enough to know merely that the sequence of domains $g_n\Omega$ does not degenerate.

5.3. **Expansion.** Before we proceed, we fix some additional terminology:

Definition 5.12. Given a properly convex domain Ω and a point $x \in \partial\Omega$, we say that a sequence $x_n \in \Omega$ *limits to x along a line L* if $x_n \to x$ in $\mathbb{RP}^{d-1}$, L is an open projective line segment $(x, x') \subseteq \Omega$, and there exists a constant $R > 0$ such that

$$d_\Omega(x_n, L) < R$$

for all n.

If the specific line L is implied (or not relevant), we will just say that x_n limits to x along a line.

If F is some face of $\partial\Omega$, we say that x_n *limits to F along a line L* if every subsequence of x_n has a subsequence limiting to some $x \in F$ along L.

Remark 5.13. If Γ is a group acting on a properly convex domain Ω, and there are $\gamma_n \in \Gamma$ so that $\gamma_n x_0$ limits to x along a line for some $x_0 \in \Omega$, the point x is often referred to as a *conical limit point* for the action of Γ on $\partial\Omega$. We will avoid this terminology, since we will need to discuss conical limit points later in a way that is not exactly equivalent.

Proposition 5.14. *Let Ω be a properly convex domain and let $\Gamma \subseteq \mathrm{Aut}(\Omega)$. Let F_- be a face of $\partial\Omega$, and let x_n be a sequence in Ω limiting to F_- along a line.*

If there exists $\gamma_n \in \Gamma$ so that $\gamma_n x_n$ is relatively compact in Ω, then:

- *(a) There exists a compact set $K \subseteq \mathrm{PGL}(d, \mathbb{R})$ such that $\gamma_n = k_n g_n$, where $k_n \in K$ and $g_n \in \mathrm{PGL}(d, \mathbb{R})$ is a sequence of $\mathcal{K}$-pseudo-loxodromics with repelling endpoint* $\mathrm{supp}(F_-)$.
- *(b) For any Riemannian metric d on $\mathrm{Gr}(k, d)$, and any $E > 1$, for all sufficiently large n there is a neighborhood U of* $\mathrm{supp}(F_-)$ *in $\mathrm{Gr}(k, d)$ such that γ_n is E-expanding (with respect to d) on U.*

Proof. Fix a compact $C \subset \Omega$ so that $\gamma_n x_n \in C$ for all n. We can move each x_n by a bounded Hilbert distance so that it lies on a fixed line segment L with an endpoint on F_-. So, by enlarging C if necessary, we can assume that the points x_n actually lie on the line L.

Let $\mathcal{K}' \subset \mathcal{C}_*(\mathbb{R}^d)$ be the compact set $\{\Omega\} \times C$. By assumption we know that for all n, we have

$$(\Omega, \gamma_n x_n) \in \mathcal{K}'.$$

Using Lemma 5.7, we can find a compact subset $\mathcal{K} \subset \mathcal{C}(\mathbb{R}^d)$ and a sequence g_n of $\mathcal{K}$-pseudo-loxodromic elements with repelling endpoint $\mathrm{supp}(F_-)$ taking x_n to x, for some $x \in \Omega \cap L$. The g_n can be chosen so that the axis contains L, implying that the set

$$\mathcal{K} \times \{x\} \subset \mathcal{C}_*(\mathbb{R}^d)$$

is compact.

Each group element $k_n = \gamma_n g_n^{-1}$ takes a pointed domain in the compact set $\mathcal{K} \times \{x\}$ to a pointed domain in the compact set $\mathcal{K}'$. But then, because $\mathrm{PGL}(d, \mathbb{R})$ acts properly on $\mathcal{C}_*(\mathbb{R}^d)$, the k_n lie in a fixed compact subset of $\mathrm{PGL}(d, \mathbb{R})$. This proves part (a).

Let V_- be the support of F_-, and let $k = \dim V_-$. The elements k_n can be viewed as lying in a compact subset of the diffeomorphisms of the compact manifold $\mathrm{Gr}(k, d)$. So, for any fixed Riemannian metric d on $\mathrm{Gr}(k, d)$, there is a constant $M > 0$ so that for all n and all $W_1, W_2 \in \mathrm{Gr}(k, d)$,

$$d(k_n W_1, k_n W_2) > M \cdot d(W_1, W_2).$$

Fix $E > 1$. Since g_n has repelling endpoint V_-, Corollary 5.10 implies that for some sufficiently large n, there is a neighborhood U of V_- in $\mathrm{Gr}(k, d)$ so that g_n satisfies

$$d(g_n W_1, g_n W_2) > \frac{E}{M} \cdot d(W_1, W_2)$$

for all $W_1, W_2 \in U$. But then we have

$$d(\gamma_n W_1, \gamma_n W_2) > E \cdot d(W_1, W_2)$$

giving us the required expansion. □

Proof of Proposition 5.1. Let Γ act cocompactly on some convex $C \subset \Omega$. Fix a Riemannian metric on $\mathrm{Gr}(k, d)$ and a constant $E > 1$.

For every face F of $\partial_i C$, there is a sequence x_n in C limiting to F along a line. Then part (b) of Proposition 5.14 implies that if $\gamma_n x_n$ is relatively compact in C for $\gamma_n \in \Gamma$, γ_n is E-expanding on a neighborhood of $\mathrm{supp}(F)$ for sufficiently large n. □

Proof of (1) $\implies$ (2) in Theorem 1.5. We apply Proposition 5.1 to $\mathrm{Cor}_\Omega(\Gamma)$, whose ideal boundary is the full orbital limit set $\Lambda_\Omega(\Gamma)$. Lemma 2.9 implies that $\Lambda_\Omega(\Gamma)$ contains all of its faces and is closed and boundary-convex, so it is the Γ-invariant subset required by the theorem. □

5.4. **North-south dynamics.** In Section 8, it will be useful to apply a consequence of part (a) of Proposition 5.14. The following can be thought of as a kind of weak version of north-south dynamics on the limit set of a group acting on a convex projective domain.

Proposition 5.15. *Let Ω be a properly convex domain, let $\Gamma \subset \mathrm{Aut}(\Omega)$, and let Λ be a closed Γ-invariant subset of $\partial\Omega$. Let F be a face of Λ, and let x_n be a sequence limiting to F along a line.*

For any sequence γ_n such that $\gamma_n x_n$ is relatively compact in Ω, there exist subspaces E_+ and E_-, with $E_+ \oplus E_- = \mathbb{R}^d$, so that:

1. *$\mathbb{P}(E_+)$, $\mathbb{P}(E_-)$ are supporting subspaces of Ω, intersecting Λ,*
2. *for every compact $K \subset \partial\Omega - \overline{F}$, a subsequence of $\gamma_n K$ converges uniformly to a subset of $\mathbb{P}(E_+)$, and a subsequence of $\gamma_n F$ converges uniformly to a subset of $\mathbb{P}(E_-)$,*
3. *for every $x \in F$ and every $z \in \partial\Omega - \overline{F}$, the sequence of line segments*

$$\gamma_n \cdot [x, z]$$

sub-converges to a line segment intersecting Ω.

We emphasize again that the subspaces $E_\pm$ above are *complementary* in $\mathbb{R}^d$. Without this additional condition, the proposition follows easily from (for example) [IZ21, Proposition 5.7].

Proof. Using Proposition 5.14, we decompose each γ_n as $k_n g_n$, for a sequence g_n of $\mathcal{K}$-pseudo-loxodromic elements with repelling endpoint $V_- = \mathrm{supp}(F)$, and k_n lying in a fixed compact in $\mathrm{PGL}(d, \mathbb{R})$. Taking a subsequence, we may assume that k_n converges to $k \in \mathrm{PGL}(d, \mathbb{R})$, so that

$$\gamma_n V_- = k_n g_n V_- = k_n V_- \to k V_-.$$

Let $E_- = kV_-$. We let V_+ be the other endpoint of the sequence g_n, let V_0 be the neutral subspace, and let $E_+ := k(V_+ \oplus V_0)$. Since Λ is closed and Γ-invariant, both $\mathbb{P}(E_+)$ and $\mathbb{P}(E_-)$ intersect Λ.

Fix a compact subset K in $\partial\Omega - F$. Proposition 5.8 implies that $g_n K$ converges uniformly to a subset of $\mathbb{P}(V_+ \oplus V_0)$. So, $k_n g_n K$ converges uniformly to a subset of $\mathbb{P}(E_+)$.

This shows parts (1) and (2). To see part (3), let L be the line segment $[x, z]$. By Proposition 2.12, we can find $R > 0$ and $x_n' \in L$ such that

$$d_\Omega(x_n, x_n') \leq R.$$

We know that $\gamma_n x_n$ lies in a fixed compact subset C of Ω. So, $\gamma_n x_n'$ lies in a closed and bounded Hilbert neighborhood of C. This is also a compact subset of Ω, so up to a subsequence, $\gamma_n x_n'$ converges to some $x_0' \in \Omega$.

The limit of the line segment $[\gamma_n x_-, \gamma_n x_n']$ is nontrivial, intersects Ω, and is a sub-segment of the limit of $[\gamma_n x_-, \gamma_n z]$, so this implies the desired result. ☐

6. Background on relative hyperbolicity

6.1. **A definition using convergence groups.** Relatively hyperbolic groups, like word-hyperbolic groups, have a wide variety of possible definitions. Here we are most interested in the dynamical properties of relatively hyperbolic groups, so we will use a dynamical characterization due to Yaman [Yam04].

Yaman's characterization uses the language of *convergence group actions*, which we review below. Convergence groups were originally studied in the context of group actions on spheres in $\mathbb{R}^d$ by Gehring and Martin [GM87], and for general group actions on compact Hausdorff spaces by Freden and Tukia [Fre97, Tuk98].

Definition 6.1. A group Γ acting on a topological space X is said to act on X as a *convergence group* if, for every sequence of distinct elements $\gamma_n \in \Gamma$, there exist (not necessarily distinct) points $a, b \in X$ and a subsequence γ_n' of γ_n such that the restriction of γ_n' to $X - \{a\}$ converges to the constant map b.

When X is a compact Hausdorff space, Γ acts on X as a convergence group if and only if Γ acts properly discontinuously on the space of *pairwise distinct triples* in X [Bow99].

Definition 6.2. Let Γ act as a convergence group on X.

- We say that $x \in X$ is a *conical limit point* if there exist *distinct* points $a, b \in X$ and an infinite sequence of elements $\gamma_n \in \Gamma$ such that $\gamma_n x \to a$ and $\gamma_n y \to b$ for all $y \neq x$ in X.
- An infinite subgroup H of Γ is a *parabolic subgroup* if it fixes a point $x \in X$ and each infinite-order element of H has exactly one fixed point in X.
- A point $x \in X$ is a *parabolic point* if its stabilizer is a parabolic subgroup. A parabolic point x is *bounded* if $\mathrm{Stab}_\Gamma(x)$ acts cocompactly on $X - \{x\}$.

When Γ acts as a convergence group on a space X with no isolated points, and every point in X is a conical limit point, we say that Γ acts as a *uniform convergence group* on X. This can be shown to be equivalent to the condition that Γ act *cocompactly* on the space of distinct triples in X [Tuk98].

An important theorem of Bowditch [Bow98] says that if Γ is a non-elementary group (i.e. not finite or virtually cyclic), Γ acts on a perfect metrizable compact space X as a uniform convergence group if and only if Γ is word-hyperbolic and X is equivariantly homeomorphic to the Gromov boundary of Γ. Yaman later proved an analogous result for relatively hyperbolic groups:

Theorem 6.3 ([Yam04]). *Let Γ be a non-elementary group, and let $\mathcal{H}$ be the collection of all conjugates of a finite collection of finitely-generated proper subgroups of Γ.*

Then Γ is hyperbolic relative to $\mathcal{H}$ if and only if Γ acts on a compact, perfect, and metrizable space X as a convergence group, every point in X is either a conical limit point or a bounded parabolic point for the Γ-action, and the parabolic points in X are exactly the fixed points of the groups in $\mathcal{H}$.

In this case, the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ is equivariantly homeomorphic to X.

We will use Theorem 6.3 as our definition of both relative hyperbolicity and the Bowditch boundary of a relatively hyperbolic group. For other definitions, see e.g. [Bow12], [DS05]. The groups in $\mathcal{H}$ are referred to as the *peripheral subgroups.*

Remark 6.4. Here we are adopting the convention that a group is hyperbolic relative to a conjugacy-closed collection of subgroups lying in finitely many conjugacy classes.

The alternative convention would be to fix a finite set $\mathcal{P} = \{P_i\}$ of representatives for these conjugacy classes, and say that the group Γ is hyperbolic relative to $\mathcal{P}$. We avoid this since we will work with the collection $\mathcal{H}$ of conjugates more often than we work with $\mathcal{P}$—the main exception is section 8.1.

7. Embedding the Bowditch boundary

Our goal here is to prove Theorem 1.9. Our first step is the following:

Proposition 7.1. *Let Ω be a properly convex domain, and let $\Gamma \subset \mathrm{Aut}(\Omega)$ be hyperbolic relative to a collection of subgroups $\mathcal{H} = \{H_i\}$ each acting convex cocompactly on Ω with disjoint full orbital limit sets $\Lambda_\Omega(H_i)$.*

Suppose Λ is a Γ-invariant subset of $\partial\Omega$ containing all of its faces and containing $\Lambda_\Omega(H_i)$ for every H_i. If $[\Lambda]_\mathcal{H}$ is the image of a Γ-equivariant embedding $\phi : \partial(\Gamma, \mathcal{H}) \to [\partial\Omega]_\mathcal{H}$, then the set

$$\Lambda_c = \Lambda - \bigcup_{H_i \in \mathcal{H}} \Lambda_\Omega(H_i)$$

contains only extreme points in $\partial\Omega$.

Proof. The equivariant homeomorphism $\phi : \partial(\Gamma, \mathcal{H}) \to [\Lambda]_\mathcal{H}$ means that Γ acts on $[\Lambda]_\mathcal{H}$ as a convergence group as in Theorem 6.3. In particular, we can classify the points of $[\Lambda]_\mathcal{H}$ as either bounded parabolic points or conical limit points, where the parabolic points are exactly the points corresponding to $\Lambda_\Omega(H_i)$.

So, if x is a point in Λ_c, it represents a conical limit point in $[\Lambda]_{\mathcal{H}}$. Suppose for a contradiction that x is not an extreme point, i.e. x lies in the interior of a nontrivial segment $[a, b] \subset \partial\Omega$. Since Λ contains all of its faces, $(a, b) \subset \Lambda$, and we can find $w, z \in \Lambda$ such that w, x, z are pairwise distinct points lying on (a, b) in that order.

Lemma 2.9 tells us that each $\Lambda_\Omega(H_i)$ contains its faces, so we know that w and z cannot lie in any $\Lambda_\Omega(H_i)$. So w, x, and z represent three distinct points in $[\Lambda]_{\mathcal{H}}$.

This means that there exist group elements $\gamma_n \in \Gamma$ so that $\gamma_n[x]_{\mathcal{H}} \to a$, and $\gamma_n[z]_{\mathcal{H}}$, $\gamma_n[w]_{\mathcal{H}}$ both converge to some $b \in [\Lambda]_{\mathcal{H}}$, with a, b distinct.

This convergence is only in $[\Lambda]_{\mathcal{H}}$. However, since the Bowditch boundary $\partial(\Gamma, \mathcal{H})$ is always compact, $[\Lambda]_{\mathcal{H}}$ is as well, and therefore its preimage Λ in the compact set $\partial\Omega$ is compact too. So, up to a subsequence, we can assume that $\gamma_n x \to u$, and $\gamma_n z \to v_1$, $\gamma_n w \to v_2$, with

$$[u]_{\mathcal{H}} = a, \ [v_1]_{\mathcal{H}} = [v_2]_{\mathcal{H}} = b.$$

The line segment $[w, z]$ must converge to the line segment $[v_1, v_2]$, which must contain u. If $v_1 = v_2$, this is clearly impossible without having $u = v_1 = v_2$. If $v_1 \neq v_2$, then v_1, v_2 both lie in $\Lambda_\Omega(H_i)$ for some H_i. Since each $\Lambda_\Omega(H_i)$ is boundary-convex (Lemma 2.9 again), u must lie in $\Lambda_\Omega(H_i)$ as well, a contradiction. $\square$

The above is important partly because of the following proposition, which we will use repeatedly in the proof of both Theorem 1.9 and its converse.

Proposition 7.2. *Let Ω be a properly convex domain, and let Λ be a boundary-convex subset of $\partial\Omega$ containing all of its faces. Let $\mathcal{H}$ be a collection of subgroups of* $\mathrm{Aut}(\Omega)$ *acting convex cocompactly with disjoint full orbital limit sets in Ω.*

If every point in $\Lambda_c = \Lambda - \bigcup_{H_i \in \mathcal{H}} \Lambda_\Omega(H_i)$ is an extreme point, then for any $x, y \in \Lambda$ with $[x]_{\mathcal{H}} \neq [y]_{\mathcal{H}}$, the segment (x, y) lies in Ω.

Proof. We will prove the contrapositive, and show that if x, y are distinct points in Λ with $(x, y) \subset \partial\Omega$, then $[x]_{\mathcal{H}} = [y]_{\mathcal{H}}$.

Assume $x, y \in \Lambda$ are distinct. Boundary-convexity tells us that if $(x, y) \subset \partial\Omega$, then $(x, y) \subset \Lambda$. Since we know Λ_c only contains extreme points, some $u \in (x, y)$ lies in $\Lambda_\Omega(H_i)$ for some $H_i \in \mathcal{H}$. Since H_i acts convex cocompactly on Ω, Lemma 2.9 implies that $[x, y]$ lies in $\Lambda_\Omega(H_i)$, which means that $[x]_{\mathcal{H}} = [y]_{\mathcal{H}}$. $\square$

The following proposition explains why we do not need to assume that Γ is discrete in the statement of Theorem 1.9.

Proposition 7.3. *If Ω, Γ, Λ are as in Theorem 1.9, and Γ is non-elementary, then Γ is discrete.*

Proof. Γ acts as a convergence group on $[\Lambda]_{\mathcal{H}}$, so it acts properly discontinuously on the space of pairwise distinct triples in $[\Lambda]_{\mathcal{H}}$, which we denote $\mathcal{T}([\Lambda]_{\mathcal{H}})$.

The map

$$\Gamma \times \mathcal{T}([\Lambda]_{\mathcal{H}}) \to \mathcal{T}([\Lambda]_{\mathcal{H}})$$

given by the Γ-action is continuous, so Γ is discrete. $\square$

We are now able to prove Theorem 1.9.

Proof of Theorem 1.9. Let Ω, Γ, Λ, $\mathcal{H}$ be as in the hypotheses for Theorem 1.9. We can assume that $\mathcal{H} \neq \{\Gamma\}$ and that Γ is infinite (if not then the theorem is trivial). This means that $\partial(\Gamma, \mathcal{H})$ contains at least two points, and Proposition 7.2 implies that $\mathrm{Cor}_\Omega(\Gamma)$ is nonempty.

If Γ is virtually infinite cyclic, the hypotheses of the theorem imply that the generator γ of a finite-index cyclic subgroup fixes a pair of points $\{x, y\}$ in $\partial\Omega$ with $(x, y) \subset \Omega$; γ acts as a translation in the Hilbert metric along the axis (x, y). This action is properly discontinuous (so Γ is discrete) and cocompact. Further, since x and y are extreme points, $\gamma^n z$ converges to either x or y as $n \to \pm\infty$ for all $z \in \Omega$, so $\Lambda_\Omega(\Gamma) = \{x, y\}$.

So we may assume Γ is non-elementary. Owing to Theorem 1.5, we only need to show that Γ is expanding at the faces of Λ; in fact we will show directly that the expansion is uniform.

Since each H_i acts convex cocompactly on Ω, Theorem 1.5 means that Γ is expanding in a neighborhood of the support of any face of $\Lambda_\Omega(H_i)$ for some H_i. In fact, we can assume that the expansion constants are uniform over all $H_i \in \mathcal{H}$ (see Remark 3.2), so we only need to consider the faces in

$$\Lambda_c = \Lambda - \bigcup_{H_i \in \mathcal{H}} \Lambda_\Omega(H_i).$$

Proposition 7.1 implies that each of these faces is actually just a point in $\partial\Omega$, whose support is equal to itself.

Let x be a point in Λ_c. We will build a sequence of points x_n in Ω limiting to x along a line (Definition 5.12), and show that the orbits $\Gamma \cdot x_n$ intersect a fixed compact set.

Since Γ is non-elementary, its Bowditch boundary contains at least three distinct conical limit points, so we can find $y, z \in \Lambda_c$ so that $[x]_\mathcal{H}, [y]_\mathcal{H}, [z]_\mathcal{H}$ are pairwise distinct.

Fix supporting hyperplanes W, V of Ω at x and z, respectively. Proposition 7.2 implies that $W \cap V$ does not contain x, y, or z, and that the line segment (x, z) is in Ω. The projective hyperplane

$$H = (W \cap V) \oplus y$$

intersects (x, z) at a point $w \in \Omega$.

Since $[x]_\mathcal{H}$ is a conical limit point, we can find a sequence $\gamma_n \in \Gamma$ so that

$$\gamma_n [x]_\mathcal{H} \to a$$

and

$$\gamma_n [z]_\mathcal{H},\ \gamma_n [y]_\mathcal{H} \to b$$

for a, b distinct. As in the proof of Proposition 7.1, we can pick subsequences so that $\gamma_n x$, $\gamma_n y$, and $\gamma_n z$ all converge to points $x_\infty, y_\infty, z_\infty$ in Λ, and $\gamma_n W$ and $\gamma_n V$ converge to supporting hyperplanes W_∞, V_∞ of Ω at x_∞ and z_∞.

Since x_∞ and z_∞ represent distinct points of $[\Lambda]_\mathcal{H}$, Proposition 7.2 implies that $W_\infty \cap V_\infty$ must not contain x_∞ or z_∞; for the same reason y_∞ is also not contained in $W_\infty \cap V_\infty$.

While $[z_\infty]_\mathcal{H} = [y_\infty]_\mathcal{H}$, it is not necessarily true that $y_\infty = z_\infty$. However, we do know that the segment (y_∞, x_∞) cannot lie in $\partial\Omega$. So, the sequence

$$\gamma_n(H \cap (x, z)) = \gamma_n w$$

cannot approach x_∞.

Proposition 7.3 means that we know Γ is discrete, and so its action on Ω is properly discontinuous. Thus $\gamma_n w$ must accumulate to an endpoint of $[x_\infty, z_\infty]$— and therefore to z_∞.

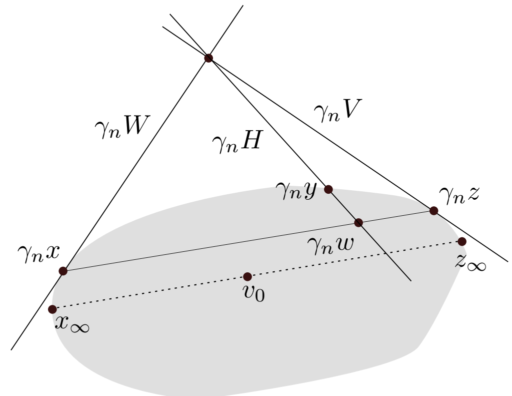

FIGURE 8. The sequence $\gamma_n w$ limits to z_∞, so the sequence $\gamma_n^{-1} v_0$ limits to x along a line.

Let ℓ be the line segment $[x, z]$. This segment has a well-defined total order, where $a < b$ if a is closer to x than b. If $\ell_n = [\gamma_n x, \gamma_n z]$, then γ_n is an order-preserving isometry from ℓ to ℓ_n, where the metric is the restricted Hilbert metric d_Ω.

Fix a basepoint v_0 on the line segment $\ell_\infty = [x_\infty, z_\infty]$, and choose $v_n \in \ell_n$ converging to v_0. Since $\gamma_n w$ converges to z_∞, we see that $v_n < \gamma_n w$ and

$$d_\Omega(v_n, \gamma_n w) \to \infty.$$

Thus we must have $\gamma_n^{-1} v_n \to x$.

But now we can apply part (b) of Proposition 5.14 to the sequence $\gamma_n^{-1} v_n \subset \ell$ to see that γ_n is eventually expanding in a neighborhood of x in $\mathbb{RP}^{d-1}$. $\square$

8. Convex cocompact groups which are relatively hyperbolic

The goal of this section is to prove Theorem 1.16.

8.1. **Non-peripheral segments in the boundary.** We start by showing that conditions (i) and (ii) of Theorem 1.16 are satisfied whenever Γ is a convex cocompact group hyperbolic relative to a collection of convex cocompact subgroups. That is, we will show:

Proposition 8.1. *Let Γ be a group hyperbolic relative to a collection $\mathcal{H}$ of subgroups, and suppose that Γ and each $H_i \in \mathcal{H}$ act on a properly convex domain Ω convex cocompactly.*

Then:

(i) The full orbital limit sets $\Lambda_\Omega(H_i)$ are disjoint for distinct $H_i, H_j \in \mathcal{H}$,
(ii) Every nontrivial segment in $\Lambda_\Omega(\Gamma)$ is contained in the full orbital limit set of some peripheral subgroup H_i,

We will closely follow the proof of a similar result of Islam and Zimmer [IZ19, Theorem 1.8 (7)]. The main idea is that a nontrivial segment ℓ in the full orbital limit set $\Lambda_\Omega(\Gamma)$ of a convex cocompact group Γ is accumulated to by segments in the boundary of some maximal properly embedded simplices in $\mathrm{Cor}_\Omega(\Gamma)$. When Γ is hyperbolic relative to a collection $\mathcal{A}$ of virtually abelian subgroups of rank at least 2, Islam and Zimmer show that $\mathcal{A}$ is in one-to-one correspondence with the set of maximal properly embedded simplices in $\mathrm{Cor}_\Omega(\Gamma)$, and then use a coset separation property due to Druţu and Sapir [DS05] to see that these maximal properly embedded simplices are isolated. This ends up implying that ℓ lies in the boundary of one of the simplices that accumulate to it.

When we do *not* assume the peripheral subgroups are virtually abelian, we need to modify this approach slightly. First, we need to assume that the peripheral subgroups act convex cocompactly on Ω ([IZ19] implies that this assumption is always satisfied in the virtually abelian case). Second, in our situation, the maximal properly embedded simplices in $\mathrm{Cor}_\Omega(\Gamma)$ do not need to be isolated. However, it is true that the convex cores $\mathrm{Cor}_\Omega(H_i)$ of the peripheral subgroups in $\mathcal{H}$ are isolated. So the desired result ends up following from the fact that every maximal properly embedded k-simplex ($k \geq 2$) in $\mathrm{Cor}_\Omega(\Gamma)$ lies in $\mathrm{Cor}_\Omega(H_i)$ for some $H_i \in \mathcal{H}$; this is Lemma 8.3 below.

8.1.1. *Cosets and convex cores of peripheral subgroups.* Let Γ be hyperbolic relative to a collection of subgroups $\mathcal{H}$, and suppose that Γ and each $H_i \in \mathcal{H}$ act convex cocompactly on a fixed properly convex domain Ω. We fix a basepoint $x \in \Omega$, and fix a finite set $\mathcal{P} = \{P_i\}$ of conjugacy representatives for $\mathcal{H}$.

The Švarc-Milnor lemma implies that Γ is finitely generated and that, under the word metric induced by any finite generating set, Γ is equivariantly quasi-isometric to the convex core $\mathrm{Cor}_\Omega(\Gamma)$ equipped with the restricted Hilbert metric d_Ω. The quasi-isometry can be taken to be the orbit map $\gamma \mapsto \gamma x$.

Since each P_i also acts convex cocompactly on Ω, each P_i is also finitely generated, and P_i is quasi-isometric to $\mathrm{Cor}_\Omega(P_i)$, which isometrically embeds into $\mathrm{Cor}_\Omega(\Gamma)$. We may assume that the quasi-isometry constants are uniform over all $P_i \in \mathcal{P}$, and fix a finite generating set for Γ containing generating sets for each P_i.

Since $g \cdot \mathrm{Cor}_\Omega(P_i) = \mathrm{Cor}_\Omega(gP_ig^{-1})$, if we fix a Γ-equivariant quasi-isometry

$$\phi : \mathrm{Cor}_\Omega(\Gamma) \to \Gamma,$$

we know ϕ restricts to a quasi-isometry

$$\mathrm{Cor}_\Omega(gP_ig^{-1}) \to gP_i,$$

with uniform quasi-isometry constants over all $g \in \Gamma$, $P_i \in \mathcal{P}$.

The cosets gP_i have a *separation property*: distinct cosets cannot stay "close" to each other over sets of large diameter. The precise statement is as follows. For any metric space X, and any $A \subseteq X$, we let

$$N_X(A; r)$$

denote the open r-neighborhood of A in X with respect to the metric d_X, and let

$$B_X(x; r)$$

denote the open r-ball about $x \in X$.

Theorem 8.2 ([DS05, Theorem 4.1 (α_1)])**.** *Let Γ be hyperbolic relative to $\mathcal{H}$, and let $\mathcal{P}$ be a finite set of conjugacy representatives. For every $r > 0$, there exists $R > 0$ such that for every distinct pair of left cosets g_1P_1, g_2P_2, the diameter of the set*

$$N_\Gamma(g_1P_1; r) \cap N_\Gamma(g_2P_2; r)$$

is at most R.

In addition, Theorem 1.7 of [DS05] implies that if $k \geq 2$, any quasi-isometrically embedded k-flat in a relatively hyperbolic group Γ is contained in the D-neighborhood of a coset gP_i of some peripheral subgroup $P_i \in \mathcal{P}$. This allows us to see the following:

Lemma 8.3. *Suppose Γ acts convex cocompactly on Ω, and that Γ is hyperbolic relative to a collection of subgroups $\mathcal{H}$ also acting convex cocompactly on Ω. Every properly embedded k-simplex ($k \geq 2$) in Ω with boundary in $\Lambda_\Omega(\Gamma)$ is contained in $\mathrm{Cor}_\Omega(H_i)$ for some $H_i \in \mathcal{H}$.*

Proof. Each such embedded k-simplex Δ is a quasi-isometrically embedded k-flat in $\mathrm{Cor}_\Omega(\Gamma)$, so $\phi(\Delta)$ is a quasi-isometrically embedded k-flat in Γ. [DS05], Theorem 1.7 implies that $\phi(\Delta)$ is contained in a uniform neighborhood of gP for some $P \in \mathcal{P}$.

Applying a quasi-inverse of ϕ tells us that Δ is in a uniform Hilbert neighborhood of $\mathrm{Cor}_\Omega(gPg^{-1})$ in Ω. So the boundary of Δ is contained in $\partial_i\mathrm{Cor}_\Omega(gPg^{-1})$, and Δ itself lies in $\mathrm{Cor}_\Omega(gPg^{-1})$. $\square$

We now quote:

Lemma 8.4 ([IZ19, Lemma 15.4])**.** *Let (u, v) be a nontrivial line segment in $\Lambda_\Omega(\Gamma)$, let $m \in (u, v)$ and $p \in \mathrm{Cor}_\Omega(\Gamma)$, and let V be the span of (u, v) and p. For any $r > 0$, $\varepsilon > 0$, there exists a neighborhood U of m in $\mathbb{P}(V)$ such that if $x \in U \cap \mathrm{Cor}_\Omega(\Gamma)$, then there is a properly embedded simplex $S_x \subset \mathrm{Cor}_\Omega(\Gamma)$ such that*

$$B_\Omega(x; r) \cap \mathbb{P}(V) \subset N_\Omega(S_x; \varepsilon).$$

Now we can prove Proposition 8.1. The proof of part (ii) is nearly identical to the proof of Lemma 15.5 in [IZ19].

Proof of Proposition 8.1. (i). Let H_i, H_j be a pair of peripheral subgroups in $\mathcal{H}$, and suppose that $\Lambda_\Omega(H_i) \cap \Lambda_\Omega(H_j)$ contains a point $x \in \partial\Omega$. We can find a pair of projective-line geodesic rays in $\mathrm{Cor}_\Omega(H_i)$ and $\mathrm{Cor}_\Omega(H_j)$ with one endpoint at x. Proposition 2.12 implies that the images of these rays have finite Hausdorff distance.

Thus, in Γ, a uniform neighborhood of the coset g_iP_i corresponding to H_i contains an infinite-diameter subset of the coset g_jP_j corresponding to H_j. So Theorem 8.2 implies that $H_i = H_j$.

(ii). Consider any nontrivial segment $[u, v]$ in $\Lambda_\Omega(\Gamma)$, and fix $m \in (u, v)$ and $p \in \mathrm{Cor}_\Omega(\Gamma)$. Theorem 8.2 implies that for some $R > 0$, there exists $r > 0$ such that the diameter of

$$N_\Omega(\mathrm{Cor}_\Omega(H_i); r) \cap N_\Omega(\mathrm{Cor}_\Omega(H_j); r)$$

is less than R whenever H_i and H_j are distinct.

Let V be the span of u, v, and p. Lemma 8.4 implies that for some neighborhood U of m in $\mathbb{P}(V)$, for every $x \in U$, there is some properly embedded simplex S_x such that

$$B_\Omega(x; R) \cap \mathbb{P}(V) \subset N_\Omega(S_x; r).$$

Lemma 8.3 means that the simplex S_x is contained in the convex hull $\mathrm{Cor}_\Omega(H_x)$ of some peripheral subgroup H_x, and part (i) implies that this peripheral subgroup is unique.

We can shrink U so that it is convex, and claim that in this case $H_x = H_y$ for all $x, y \in U \cap \mathrm{Cor}_\Omega(\Gamma)$. By convexity, it suffices to show this when $d_\Omega(x, y) \leq R/2$. Then

$$B_\Omega(x; R/2) \cap \mathbb{P}(V) \subset B_\Omega(y; R) \cap \mathbb{P}(V) \subset N_\Omega(S_y; r)$$

so the diameter of

$$N_\Omega(\mathrm{Cor}_\Omega(H_x); r) \cap N_\Omega(\mathrm{Cor}_\Omega(H_y); r)$$

is at least the diameter of $B_\Omega(x; R/2) = R$. Thus $H_x = H_y$.

Fix $H = H_x$ for some $x \in U \cap \mathrm{Cor}_\Omega(\Gamma)$. Then, if x_n is a sequence in $\mathrm{Cor}_\Omega(\Gamma)$ approaching m, there is a sequence $x_n' \in \mathrm{Cor}_\Omega(H)$ such that

$$d_\Omega(x_n, x_n') \leq k,$$

for k independent of n. Up to a subsequence, x_n' converges to some $x' \in \Lambda_\Omega(H)$. Proposition 2.12 implies that

$$F_\Omega(x') = F_\Omega(m) \supseteq (u, v).$$

$\Lambda_\Omega(H)$ contains x'. It is also closed and contains all of its faces (Lemma 2.9). So $[u, v] \subset \Lambda_\Omega(H)$.

□

8.2. **Convex cocompact and no relative segment implies relatively hyperbolic.** We now turn to the rest of Theorem 1.16. As in our proof of Theorem 1.9, the main tool will be Yaman's dynamical characterization of relative hyperbolicity (Theorem 6.3). If Γ is virtually cyclic, Yaman's theorem does not apply, but in this case Γ is hyperbolic and the result follows from [DGK17].

Throughout the rest of this section, we assume (as in the hypotheses to Theorem 1.16) that Ω is a properly convex domain in $\mathbb{RP}^{d-1}$ preserved by a discrete non-elementary group Γ acting convex cocompactly with full orbital limit set $\Lambda_\Omega(\Gamma)$, and $\mathcal{H}$ is a conjugacy-invariant set of subgroups of Γ lying in finitely many conjugacy classes, with each $H_i \in \mathcal{H}$ acting convex cocompactly on Ω. We also assume $\mathcal{H} \neq \{\Gamma\}$, since the result is trivial in this case.

We will prove the following:

Proposition 8.5. *Suppose that conditions (i), (ii), and (iii) of Theorem 1.16 hold for the collection of subgroups $\mathcal{H}$. Then:*

1. *Γ acts as a convergence group on $[\Lambda_\Omega(\Gamma)]_\mathcal{H}$,*
2. *$[\Lambda_\Omega(\Gamma)]_\mathcal{H}$ is compact, metrizable, and perfect,*
3. *the groups H_i are parabolic subgroups, and their fixed points are bounded parabolic,*
4. *every point in*

$$[\Lambda_\Omega(\Gamma)]_\mathcal{H} - \{[\Lambda_\Omega(H_i)]_\mathcal{H} : H_i \in \mathcal{H}\}$$

is a conical limit point for the Γ-action on $[\Lambda_\Omega(\Gamma)]_\mathcal{H}$.

Since convex cocompact groups are always finitely generated, Theorem 1.16 is a direct consequence of Proposition 8.5, Proposition 8.1, and Theorem 6.3.

8.2.1. *Dynamics of the* Γ*-action on* $\Lambda_\Omega(\Gamma)$. We start by establishing a basic dynamical fact about the action of Γ on $\Lambda_\Omega(\Gamma)$ and $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$. We need to recall some basic properties of *divergent sequences* (that is, sequences which leave every compact) in $\mathrm{PGL}(d, \mathbb{R})$.

Definition 8.6. Let g_n be a divergent sequence of elements in $\mathrm{PGL}(d, \mathbb{R})$. We say that a pair of nontrivial subspaces $E_+, E_- \subset \mathbb{R}^d$ is a *pair of attracting and repelling subspaces* for g_n (or just an *attracting/repelling pair*) if $\dim(E_+) + \dim(E_-) = d$, and there is a subsequence g_m of g_n so that for any compact $K \subset \mathbb{RP}^{d-1} - \mathbb{P}(E_-)$, the set $g_m K$ accumulates uniformly on $\mathbb{P}(E_+)$.

Note that neither the subspaces E_+, E_- nor even their respective dimensions are uniquely determined by the sequence g_n itself. We also emphasize that while the subspaces in an attracting/repelling pair have complementary dimension, they do *not* need to be transverse.

It is always possible to find at least one pair of attracting and repelling subspaces for a divergent sequence $g_n \in \mathrm{PGL}(d, \mathbb{R})$, for instance by embedding $\mathrm{PGL}(d, \mathbb{R})$ into the compact space $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$. Then, if $g \in \mathbb{P}(\mathrm{End}(\mathbb{R}^d))$ is any accumulation point of g_n, the image and kernel of g form a pair of attracting and repelling subspaces for the sequence. However, g_n may have other attracting/repelling pairs which do not arise in this way. For example, if g_n is given by the sequence of matrices

$$\begin{pmatrix} 2^n & & \\ & 1 & \\ & & 2^{-n} \end{pmatrix},$$

then the limit of g_n in $\mathbb{P}(\mathrm{End}(\mathbb{R}^3))$ is the (projectivized) matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, yielding the attracting subspace $\langle e_1 \rangle$ and repelling subspace $\langle e_2, e_3 \rangle$. But in this example, the subspace $\langle e_1, e_2 \rangle$ is also an attracting subspace, paired with the repelling subspace $\langle e_3 \rangle$.

The result below is certainly well-known, but we include a proof for completeness.

Lemma 8.7. *Let* Ω *be a properly convex domain in* $\mathbb{RP}^{d-1}$*, let* Γ *be a subgroup of* $\mathrm{Aut}(\Omega)$*, and let* Λ *be any closed* Γ*-invariant subset of* $\partial\Omega$ *with nonempty convex hull in* Ω*.*

If E_+ *and* E_- *are a pair of attracting and repelling subspaces for some divergent sequence* $\{\gamma_n\} \subset \Gamma$*, then* $\mathbb{P}(E_+)$ *and* $\mathbb{P}(E_-)$ *are supporting subspaces of* Ω *that intersect* Λ *nontrivially.*

Note that if E_+ and E_- are attracting and repelling subspaces arising from a limit of g_n in $\mathbb{P}(\mathrm{End}(\mathbb{R}^d))$, then this lemma can be seen as a consequence of [IZ21, Proposition 5.6] (and in fact this case of the lemma is already strong enough for our intended application).

Proof. It suffices to show the claim for E_+, because replacing γ_n with γ_n^{-1} reverses the role of the attracting and repelling subspaces.

Since Ω is open, it is not contained in $\mathbb{P}(E_-)$. So, for some $x \in \Omega$, the limit of $\gamma_n x$ is contained in $\mathbb{P}(E_+)$. Since Ω is Γ-invariant, $\mathbb{P}(E_+)$ intersects $\overline{\Omega}$ nontrivially.

Let E_+^* be the subspace of $(\mathbb{R}^d)^*$ consisting of functionals which vanish on E_+. Then E_+^* is an attracting subspace for the sequence γ_n under the dual action of Γ on $(\mathbb{R}^d)^*$. So, by the previous argument, $\mathbb{P}(E_+^*)$ intersects $\overline{\Omega^*}$ nontrivially, which means $\mathbb{P}(E_+)$ cannot intersect Ω.

This shows that $\mathbb{P}(E_+)$ is a supporting subspace of Ω (and therefore $\mathbb{P}(E_-)$ is as well). To see that $\mathbb{P}(E_+)$ intersects Λ nontrivially, note that since Λ has nonempty convex hull in Ω and $\mathbb{P}(E_-)$ is a supporting subspace of Ω, Λ is not a subset of $\mathbb{P}(E_-) \cap \partial\Omega$. So, for some $x \in \Lambda$, $\gamma_n x$ accumulates to a point y in $\mathbb{P}(E_+)$; since Λ is Γ-invariant and closed, y is in Λ also. ☐

A straightforward consequence of Lemma 8.7 is part (1) of Proposition 8.5:

Proposition 8.8. *Γ acts as a convergence group on $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.*

Proof. Let γ_n be an infinite sequence in Γ. Since Γ is discrete, γ_n is divergent, so we let E_+, E_- be (the projectivizations of) a pair of attracting and repelling subspaces, and extract a subsequence so that for any compact $K \subset \mathbb{RP}^{d-1} - E_-$, $\gamma_n K$ converges to a subset of E_+.

Lemma 8.7 implies that E_+ and E_- are both supporting subspaces of Ω and both intersect $\Lambda_\Omega(\Gamma)$ nontrivially. The intersections $E_+ \cap \Lambda_\Omega(\Gamma)$ and $E_- \cap \Lambda_\Omega(\Gamma)$ are respectively the closures of subsets of a pair of faces $F_+, F_- \subset \Lambda_\Omega(\Gamma)$. By assumption, every face in $\Lambda_\Omega(\Gamma)$ containing a nontrivial projective segment lies in some $\Lambda_\Omega(H_i)$, so each face in $\Lambda_\Omega(\Gamma)$ represents a single point of $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$. So we have

$$[E_- \cap \Lambda_\Omega(\Gamma)]_{\mathcal{H}} = a, \quad [E_+ \cap \Lambda_\Omega(\Gamma)]_{\mathcal{H}} = b$$

for (not necessarily distinct) points $a, b \in [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.

Let $[K]_{\mathcal{H}}$ be a compact subset of $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}} - \{a\}$, where K is the preimage of $[K]_{\mathcal{H}}$ in $\Lambda_\Omega(\Gamma)$. Because $\Lambda_\Omega(\Gamma)$ is compact, so is K. Moreover, K cannot intersect E_-. So, $\gamma_n \cdot K$ converges to a subset of $E_+ \cap \Lambda_\Omega(\Gamma)$, and $\gamma_n[K]_{\mathcal{H}}$ converges to b. ☐

8.2.2. *Topological properties of $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.* Next, we will check that $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ satisfies each of the properties in part (2) of Proposition 8.5. The first, compactness, is immediate from the compactness of $\Lambda_\Omega(\Gamma)$.

Showing that $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ is metrizable is equivalent to showing that it is Hausdorff, since it is a quotient of a compact metrizable space.

Let

$$\pi_{\mathcal{H}} : \Lambda_\Omega(\Gamma) \to [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$$

be the quotient map. We will show that if a is a point in $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$, then we can find arbitrarily small open neighborhoods of $\pi_{\mathcal{H}}^{-1}(a)$ in $\Lambda_\Omega(\Gamma)$ which are of the form $\pi_{\mathcal{H}}^{-1}(U)$ for $U \subset [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.

Our first step is the following:

Lemma 8.9. *Fix any metric $d_{\mathbb{P}}$ on projective space. Let $a \in [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.*

For any $\varepsilon > 0$, there exists a subset $W(a, \varepsilon) \subset \Lambda_\Omega(\Gamma)$ satisfying:

1. *$W(a, \varepsilon) = \pi_{\mathcal{H}}^{-1}(V)$ for some $V \subset [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$,*
2. *$W(a, \varepsilon)$ contains an open neighborhood of $\pi_{\mathcal{H}}^{-1}(a)$ in $\Lambda_\Omega(\Gamma)$, and*
3. *For every $z \in W(a, \varepsilon)$, we have*

$$d_{\mathbb{P}}(z, \pi_{\mathcal{H}}^{-1}(a)) < \varepsilon.$$

Proof. Let $X_a = \pi_{\mathcal{H}}^{-1}(a)$. For any open set U in $\Lambda_\Omega(\Gamma)$ containing X_a, we let $W(U)$ be the set

$$\pi_{\mathcal{H}}^{-1}([U]_{\mathcal{H}}) = U \cup \{x \in \Lambda_\Omega(H_i) : \Lambda_\Omega(H_i) \cap U \neq \emptyset\}.$$

$W(U)$ is a subset of $\Lambda_\Omega(\Gamma)$ satisfying conditions (1) and (2). We claim that for any given $\varepsilon > 0$, $W(U)$ also satisfies condition (3) as long as U is sufficiently small.

We proceed by contradiction. Suppose otherwise, so that there is some $\varepsilon > 0$ so that for a shrinking sequence of open neighborhoods U_n of X_a, there is some $H_n \in \mathcal{H}$ such that

$$\Lambda_\Omega(H_n) \cap U_n \neq \emptyset,$$

and $\Lambda_\Omega(H_n)$ contains a point z_n such that $d_{\mathbb{P}}(z_n, X_a) \geq \varepsilon$.

We write $\Lambda_n = \Lambda_\Omega(H_n)$. We can choose a subsequence so that in the topology on nonempty closed subsets of projective space, Λ_n converges to some closed subset of $\Lambda_\Omega(\Gamma)$, which we denote Λ_∞, and z_n converges to $z_\infty \in \Lambda_\infty$ such that $d_{\mathbb{P}}(z_\infty, X_a) \geq \varepsilon$.

The set Λ_∞ intersects every open subset of $\Lambda_\Omega(\Gamma)$ containing X_a, and since X_a is a closed subset of a metrizable space, this means Λ_∞ intersects X_a. We will get a contradiction by showing that in fact $z_\infty \in X_a$.

We consider two cases:

Case 1: $\mathrm{Hull}_\Omega(\Lambda_\infty)$ *is nonempty.* Since the groups in $\mathcal{H}$ lie in only finitely many conjugacy classes, up to a subsequence, the H_n are all conjugate to each other, and we may assume that $\Lambda_n = \gamma_n \Lambda_0$ for a sequence $\gamma_n \in \Gamma$.

We can find a sequence $x_n \in \mathrm{Hull}_\Omega(\Lambda_n)$ converging to some $x_\infty \in \mathrm{Hull}_\Omega(\Lambda_\infty)$. Since the action of H_0 on $\mathrm{Hull}_\Omega(\Lambda_0)$ is cocompact, there is some fixed $R > 0$ so that every H_0-orbit in $\mathrm{Hull}_\Omega(\Lambda_0)$ intersects the Hilbert ball of radius R about x_0. Since H_n is a conjugate of H_0 by an isometry of the Hilbert metric on Ω, the same is true (with the same R) for every x_n, H_n, and Λ_n.

So, we can find a sequence

$$\mu_n \in \gamma_n H_n \gamma_n^{-1}$$

so that $\mu_n \gamma_n x_0$ lies in the Hilbert ball of radius R about x_n. Since x_n converges to $x_\infty \in \Omega$, and Γ acts properly discontinuously on Ω, this means that a subsequence of $\mu_n \gamma_n$ is eventually constant. Because $\mu_n \gamma_n \Lambda_0 = \gamma_n \Lambda_0$, this means we can assume there is some fixed $\gamma \in \Gamma$ so that

$$\Lambda_\infty = \gamma \Lambda_0 = \Lambda_\Omega(\gamma H_0 \gamma^{-1}).$$

But then since the limit sets $\Lambda_\Omega(H_i)$ are disjoint, we must have $X_a = \Lambda_\infty$, which means $z_\infty \in X_a$.

Case 2: $\mathrm{Hull}_\Omega(\Lambda_\infty)$ *is empty.* In this case, $\mathrm{Hull}_\Omega(\Lambda_\infty)$ must be contained in the closure of some face F of $\partial\Omega$. We may choose this face minimally, which means that $\mathrm{Hull}_\Omega(\Lambda_\infty)$ is not contained in ∂F. Then, because $\Lambda_\Omega(\Gamma)$ is boundary convex, we know that $\Lambda_\Omega(\Gamma) \cap F$ is nonempty. Because $\Lambda_\Omega(\Gamma)$ contains all of its faces this means that $F \subset \Lambda_\Omega(\Gamma)$.

If F is a single point, then its closure is also a singleton, hence Λ_∞ is the singleton $\{z_\infty\}$. So in this case z_∞ lies in X_a. If F is *not* a single point, it contains a nontrivial segment. By assumption, this segment lies in $\Lambda_\Omega(H_i)$ for some H_i; since $\Lambda_\Omega(H_i)$ is closed and contains its faces, all of $\overline{F}$ lies in $\Lambda_\Omega(H_i)$ as well. But then $\Lambda_\Omega(H_i)$ intersects both X_a and z_∞. Since $X_a = \pi_{\mathcal{H}}^{-1}(a)$ we must have $X_a = \Lambda_\Omega(H_i) = [z_\infty]_{\mathcal{H}}$ and therefore $z_\infty \in X_a$ in this case as well.

$\square$

Proposition 8.10. $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ *is Hausdorff.*

Proof. Let a, a' be distinct points in $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$, and let X_a, X_a' be the preimages of a and a' in $\Lambda_\Omega(\Gamma)$.

Since X_a and X'_a are closed disjoint subsets of the metrizable space $\Lambda_\Omega(\Gamma)$, there is some $\varepsilon > 0$ such that for any $x \in X_a$, $x' \in X'_a$,

$$d(x, x') > 2\varepsilon.$$

For each $n \in \mathbb{N}$, we define a sequence of sets U_n containing X_a as follows. We let $U_0 = X_a$. Then, for each $n > 0$, we take U_n to be the set

$$\bigcup_{b \in [U_{n-1}]_{\mathcal{H}}} W(b, \varepsilon/2^n),$$

where $W(b, \varepsilon/2^n)$ is the set given by Lemma 8.9. Note that each U_n is a set of the form $\pi_{\mathcal{H}}^{-1}(V)$ for some $V \subset [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$; moreover, if $z \in U_n$, then

$$d(z, U_{n-1}) < \varepsilon/2^n.$$

Consider the set $U = \bigcup_{n \in \mathbb{N}} U_n$. This set is the preimage of some $V \subset [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$, and it must be contained in an ε-neighborhood of X_a. In addition, U is open in $\Lambda_\Omega(\Gamma)$: if z is in U_n, then U_{n+1} contains $W([z]_{\mathcal{H}}, \varepsilon/2^{n+1})$, which in turn contains an open neighborhood of z.

This means that $[U]_{\mathcal{H}}$ is an open set in $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ containing a. We can similarly construct an open set $[U']_{\mathcal{H}}$ containing a' such that U' is contained in an ε-neighborhood of X'_a. We know U and U' are disjoint, so $[U]_{\mathcal{H}}$ and $[U']_{\mathcal{H}}$ separate a and a'. $\square$

Next we show that the space $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ is perfect, i.e. $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ contains no isolated points.

Proposition 8.11. *$[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ is perfect.*

Proof. Fix $a \in [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ and a representative x of a. Let $F = F_\Omega(x)$.

Let x_n be a sequence of points in $\mathrm{Cor}_\Omega(\Gamma)$ converging to x in $\mathbb{RP}^{d-1}$. Convex cocompactness means that for some $\gamma_n \in \Gamma$, $\gamma_n^{-1} x_n \in C$ for a fixed compact $C \subset \Omega$.

This means that (up to a subsequence) for fixed $x_0 \in \Omega$, $\gamma_n x_0$ converges to a point in F. And since γ_n acts by Hilbert isometries, Proposition 2.12 implies that if B is any open ball with finite Hilbert radius about x_0, $\gamma_n B$ converges uniformly to a subset of F.

γ_n is divergent in $\mathrm{PGL}(d, \mathbb{R})$, so let E_+ and E_- be a pair of attracting and repelling (projective) subspaces for the sequence γ_n. We know that E_+ and E_- are supporting subspaces of Ω, and that

$$[E_- \cap \Lambda_\Omega(\Gamma)]_{\mathcal{H}}, \quad [E_+ \cap \Lambda_\Omega(\Gamma)]_{\mathcal{H}}$$

are single points in $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$. Moreover, since an open subset of projective space converges under γ_n to F, E_+ intersects F, and $[E_+ \cap \Lambda_\Omega(\Gamma)]_{\mathcal{H}} = a$. Let $b = [E_- \cap \Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.

Since we assume $\mathcal{H} \neq \{\Gamma\}$, $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ cannot be a single point, and since Γ is non-elementary, $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ contains at least three points. So, we can find a pair of points $c_1, c_2 \in [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ such that $\{b, c_1, c_2\}$ are pairwise distinct. Both c_1 and c_2 have a representative which does not lie in E_-, so both $\gamma_n c_1$ and $\gamma_n c_2$ converge to a; since $c_1 \neq c_2$, a cannot be isolated. $\square$

8.2.3. *Parabolic points in* $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$. Our next task is to verify part (3) of Proposition 8.5—that is, to show that points stabilized by our candidate peripheral subgroups are bounded parabolic points.

Proposition 8.12. *Each point* $[\Lambda_\Omega(H_i)]_{\mathcal{H}}$ *in* $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ *is a parabolic point for the action of* Γ*, with stabilizer* H_i*.*

Proof. The fact that H_i is self-normalizing implies that H_i is exactly the stabilizer of $[\Lambda_\Omega(H_i)]_{\mathcal{H}}$ in Γ: for general $g \in \mathrm{Aut}(\Omega)$,

$$g \cdot \Lambda_\Omega(H_i) = \Lambda_\Omega(gH_ig^{-1}),$$

and since we assume that the full orbital limit sets of distinct groups in $\mathcal{H}$ are disjoint, $g \in \Gamma$ preserves $\Lambda_\Omega(H_i)$ if and only if g normalizes H_i.

So we just need to check that the groups H_i are parabolic. Let $\gamma \in H_i$ be an infinite-order element, so that γ^n is a divergent sequence in $\mathrm{PGL}(d, \mathbb{R})$. We want to show that γ does not fix any point in $[\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$ other than $[\Lambda_\Omega(H_i)]_{\mathcal{H}}$.

Let E_+ and E_- be a pair of attracting and repelling projective subspaces for the sequence γ^n. Lemma 8.7 implies that both E_+ and E_- support Ω and intersect $\Lambda_\Omega(H_i)$ nontrivially.

Let $b \in [\Lambda_\Omega(\Gamma)]_{\mathcal{H}} - \{[\Lambda_\Omega(H_i)]_{\mathcal{H}}\}$, let $y \in \Lambda_\Omega(H_i) \cap E_-$, and let $x \in \Lambda_\Omega(\Gamma)$ be a representative of b. Proposition 7.2 implies that x cannot lie in E_-, so $\gamma^n x$ converges to a point in $\Lambda_\Omega(\Gamma) \cap E_+$. Then $\gamma^n b$ converges to $[\Lambda_\Omega(H_i)]_{\mathcal{H}}$, and in particular γ does not fix b. $\square$

We still need to show that the parabolic points $[\Lambda_\Omega(H_i)]_{\mathcal{H}}$ are *bounded* parabolic points, i.e. that H_i acts cocompactly on

$$[\Lambda_\Omega(\Gamma)]_{\mathcal{H}} - \{[\Lambda_\Omega(H_i)]_{\mathcal{H}}\}.$$

Our strategy is to show that the set

$$\Lambda_i = \Lambda_\Omega(\Gamma) - \Lambda_\Omega(H_i)$$

is a closed subset of the interior of some convex open set Ω_{H_i}, such that the ideal boundary of Λ_i in Ω_{H_i} is exactly $\Lambda_\Omega(H_i)$. Then, we can use the fact that H_i is uniformly expanding in supports at $\Lambda_\Omega(H_i)$ to see that the action of H_i on Λ_i is cocompact.

If H_i is irreducible (or more generally, if we know that H_i contains a *proximal element*), then as a consequence of [Ben00, Proposition 3.1] (or [DGK17, Proposition 4.5]), we can simply take Ω_{H_i} to be the unique H_i-invariant *maximal* properly convex domain $\Omega_{\max}$ in $\mathbb{RP}^{d-1}$. Since we do not know if H_i contains a proximal element in general, we do not know if such a maximal domain exists. So, we will construct Ω_{H_i} directly.

To do so, we consider the *dual* full orbital limit set $\Lambda_{\Omega^*}(\Gamma)$ of a group Γ acting on a properly convex domain Ω. i.e. the full orbital limit set in Ω^* of Γ viewed as a subgroup of $\mathrm{Aut}(\Omega^*)$. Each element of $\Lambda_{\Omega^*}(\Gamma)$ is an element of $\partial\Omega^*$, i.e. a supporting hyperplane of Ω.

Proposition 8.13. *Let* Γ *be any subgroup of* $\mathrm{Aut}(\Omega)$*.*

1. *For every* $x \in \Lambda_\Omega(\Gamma)$ *there exists* $w \in \Lambda_{\Omega^*}(\Gamma)$ *such that* $w(x) = 0$*.*
2. *For every* $w \in \Lambda_{\Omega^*}(\Gamma)$ *there exists* $x \in \Lambda_\Omega(\Gamma)$ *such that* $w(x) = 0$*.*

The statement follows from e.g. Proposition 5.6 in [IZ21]; we provide an alternative proof for convenience.

Proof. The two statements are dual to each other, so we only need to prove (1).

Given a point $x \in \Omega$, and $W \in \Omega^*$, we consider a quantity

$$\delta_\Omega(x, W)$$

defined in [DGK17] as follows:

$$\delta_\Omega(x, W) = \inf_{z \in W} \{\min\{[a_z, x; b_z, z], [b_z, x; a_z, z]\},$$

where a_z and b_z are the points in $\partial\Omega$ such that a_z, x, b_z, z lie on a projective line. The function $\delta_\Omega(x, W)$ can be thought of as an $\mathrm{Aut}(\Omega)$-invariant measure of how "close" x is to $\partial\Omega$, relative to the projective hyperplane W: it takes on nonzero values for $x \in \Omega$, $W \in \Omega^*$, and for fixed $W \in \Omega^*$ and x_n converging to $\partial\Omega$, $\delta_\Omega(x_n, W)$ converges to 0.

We now take $z \in \Lambda_\Omega(\Gamma)$, and choose $\gamma_n \in \Gamma$, $z_0 \in \Omega$ so that $\gamma_n z_0 \to z$. Fix some $W_0 \in \Omega^*$, and consider the sequence $\gamma_n W_0$. Up to a subsequence, this converges to some $W \in \Lambda^*_\Omega(\Gamma)$.

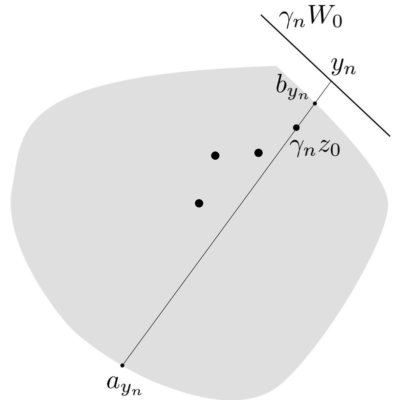

FIGURE 9. If $\gamma_n z_0$ approaches the boundary of Ω, and $\delta_\Omega(\gamma_n z_0, \gamma_n W_0)$ is bounded away from 0, $\gamma_n W_0$ must limit to a hyperplane containing the limit of $\gamma_n z_0$.

Since $\delta_\Omega(x, W)$ is Γ-invariant, for any sequence

$$y_n \in \gamma_n W_0,$$

both of the cross-ratios

$$[a_{y_n}, \gamma_n z_0; b_{y_n}, y_n], [b_{y_n}, \gamma_n z_0; a_{y_n}, y_n]$$

remain bounded away from 0 as $n \to \infty$. But since $\gamma_n z_0$ approaches $z \in \partial\Omega$, we can choose y_n so that exactly one of a_{y_n}, b_{y_n} also approaches z. Thus, y_n approaches z as well, and so W contains z. $\square$

Next, we consider the *dual convex core* for the Γ-action on Ω.

Definition 8.14. Let $\Omega \subset \mathbb{RP}^{d-1}$ be a properly convex domain, and let $\Gamma \subseteq \mathrm{Aut}(\Omega)$. The *dual convex core* $\mathrm{Cor}^*_\Omega(\Gamma)$ is the convex set

$$[\mathrm{Hull}_{\Omega^*}(\Lambda_{\Omega^*}(\Gamma))]^*.$$

Equivalently, $\mathrm{Cor}^*_\Omega(\Gamma)$ is the unique connected component of

$$\mathbb{RP}^{d-1} - \bigcup_{W \in \Lambda_{\Omega^*}(\Gamma)} W$$

which contains Ω.

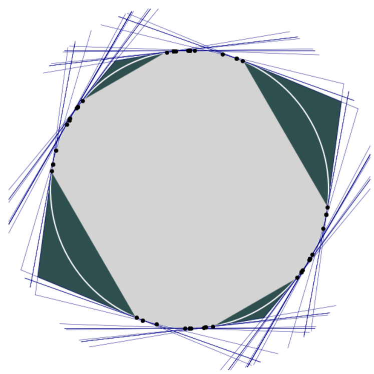

FIGURE 10. Part of the limit set and dual limit set for a group Γ acting convex compactly on the projective model for $\mathbb{H}^2$ (the interior of the white circle). $\mathrm{Cor}_\Omega(\Gamma)$ is the light region, and $\mathrm{Cor}^*_\Omega(\Gamma)$ is the dark region.

As long as $\Lambda_{\Omega^*}(\Gamma)$ contains at least two points, $\mathrm{Cor}^*_\Omega(\Gamma)$ does not contain all of $\mathbb{RP}^{d-1}$. It can be viewed as an intersection of convex subspaces, so it is convex in the sense of Definition 2.2, but in general it is not properly convex.

We can use the dual convex core to finish proving part (3) of Proposition 8.5.

Proposition 8.15. *The stabilizer of* $[\Lambda_\Omega(H_i)]_\mathcal{H}$ *acts cocompactly on*

$$\Lambda_i = [\Lambda_\Omega(\Gamma)]_\mathcal{H} - \{[\Lambda_\Omega(H_i)]_\mathcal{H}\}.$$

Proof. Let $\Omega_{H_i} = \mathrm{Cor}^*_\Omega(H_i)$ be the dual convex core of H_i in Ω. Proposition 8.13 implies that $\Lambda_\Omega(H_i)$ lies in the boundary of Ω_{H_i}.

Moreover, the set $\Lambda_\Omega(\Gamma) - \Lambda_\Omega(H_i)$ lies in the interior of Ω_{H_i}—for, every point in the boundary of Ω_{H_i} is contained in a projective hyperplane $W \in \Lambda_{\Omega^*}(H_i)$, and each such hyperplane supports some $x \in \Lambda_\Omega(H_i)$. Since W is also a supporting hyperplane of Ω, Proposition 7.2 implies that no $y \in \Lambda_\Omega(\Gamma) - \Lambda_\Omega(H_i)$ lies in W.

$\Lambda_\Omega(\Gamma)$ is thus a closed subset of Ω_{H_i} whose ideal boundary in Ω_{H_i} is contained in $\Lambda_\Omega(H_i)$. Since H_i acts convex cocompactly on Ω, it is uniformly expanding in supports at $\Lambda_\Omega(H_i)$ by Theorem 1.5. Then Proposition 3.4 (applied to the convex domain Ω_{H_i}) implies that the action of H_i on $\Lambda_\Omega(\Gamma) - \Lambda_\Omega(H_i)$ is cocompact—which means that the H_i-action on the quotient $[\Lambda_\Omega(\Gamma)]_\mathcal{H} - \{[\Lambda_\Omega(H_i)]_\mathcal{H}\}$ is cocompact as well. $\square$

8.2.4. *Conical limit points in* $[\Lambda_\Omega(\Gamma)]_\mathcal{H}$. Finally we check part (4) of Proposition 8.5—that the remaining points in our candidate Bowditch boundary are indeed conical limit points. We will do this in two steps.

Lemma 8.16. *Let $H_i \in \mathcal{H}$, let*
$$x_n \in \Lambda_\Omega(\Gamma) - \Lambda_\Omega(H_i)$$
be a sequence approaching $x \in \Lambda_\Omega(H_i)$, and let $F = F_\Omega(x)$. If h_n is a sequence such that $h_n[x_n]_{\mathcal{H}}$ is relatively compact in
$$[\Lambda_\Omega(\Gamma)]_{\mathcal{H}} - \{[\Lambda_\Omega(H_i)]_{\mathcal{H}}\},$$
then for any compact
$$K \subset \Lambda_\Omega(\Gamma) - \overline{F},$$
h_n sub-converges on $[K]_{\mathcal{H}}$ to the constant map $[\Lambda_\Omega(H_i)]_{\mathcal{H}}$.

Proof. Any such sequence h_n must be divergent, so we let E_+ and E_- be a pair of attracting and repelling projective subspaces for h_n. We know from Lemma 8.7 that E_+ and E_- are supporting subspaces of Ω, each intersecting $\Lambda_\Omega(H_i)$ nontrivially. Proposition 7.2 implies that $E_+ \cap \Lambda_\Omega(\Gamma) \subset \Lambda_\Omega(H_i)$ and $E_- \cap \Lambda_\Omega(\Gamma) \subset \Lambda_\Omega(H_i)$. We can see that the subspace E_- must contain x, since otherwise $h_n x_n$ would subconverge to a point in $E_+ \cap \partial\Lambda_\Omega(\Gamma) \subseteq \Lambda_\Omega(H_i)$.

But then $E_- \cap \partial\Omega$ is a subset of $\overline{F}$. Then, since E_+ and E_- are a pair of attracting and repelling subspaces, if K is compact in $\Lambda_\Omega(\Gamma) - E_-$, we know $h_n K$ uniformly accumulates on $E_+ \cap \Lambda_\Omega(\Gamma) \subset \Lambda_\Omega(H_i)$. ☐

Proposition 8.17. *Every element of the set*
$$[\Lambda_\Omega(\Gamma)]_{\mathcal{H}} - \{[\Lambda_\Omega(H_i)]_{\mathcal{H}} : H_i \in \mathcal{H}\}$$
is a conical limit point for the action of Γ on $\Lambda_\Omega(\Gamma)$.

Proof. By assumption, any point in this set has a unique representative $x \in \Lambda_\Omega(\Gamma)$ which is an extreme point in $\partial\Omega$. Fix a sequence $x_n \in \Omega$ limiting to x along a line, and let $\gamma_n \in \Gamma$ be group elements taking x_n back to some fixed compact in Ω.

Proposition 5.15 implies that there is a supporting subspace E_+ of Ω, intersecting $\Lambda_\Omega(\Gamma)$, so that $\gamma_n x$ limits to some $x' \in \Lambda_\Omega(\Gamma)$ not intersecting E_+, and if K is any compact subset of $\Lambda_\Omega(\Gamma) - x$, a subsequence of $\gamma_n K$ converges uniformly to a subset of $E_+ \cap \Lambda_\Omega(\Gamma)$. In particular, γ_n converges uniformly on compacts in
$$[\Lambda]_{\mathcal{H}} - \{[x]_{\mathcal{H}}\}$$
to the constant map $[E_+ \cap \Lambda_\Omega(\Gamma)]_{\mathcal{H}}$.

If $[x']_{\mathcal{H}} \neq [E_+ \cap \Lambda_\Omega(\Gamma)]_{\mathcal{H}}$, then we are done. However, it is also possible that x' and $E_+ \cap \Lambda_\Omega(\Gamma)$ both lie in the same full orbital limit set of some convex cocompact subgroup H_i.

In this case, we use the fact that $[\Lambda_\Omega(H_i)]_{\mathcal{H}}$ is a bounded parabolic fixed point (Proposition 8.15) to find a sequence $h_n \in H_i$ such that $h_n \cdot [\gamma_n x]_{\mathcal{H}}$ lies in a fixed compact set C in
$$[\Lambda_\Omega(\Gamma)]_{\mathcal{H}} - \{[\Lambda_\Omega(H_i)]_{\mathcal{H}}\},$$
and consider the sequence of group elements $h_n\gamma_n$. We will show that $h_n\gamma_n$ is a conical limit sequence for x, i.e. that after taking a subsequence, for distinct $a, b \in [\Lambda_\Omega(\Gamma)]_{\mathcal{H}}$, we have $h_n\gamma_n[x]_{\mathcal{H}} \to a$ and $h_n\gamma_n[K]_{\mathcal{H}} \to b$ for any compact $[K]_{\mathcal{H}} \subset [\Lambda_\Omega(\Gamma)]_{\mathcal{H}} - [x]_{\mathcal{H}}$.

So, fix an arbitrary compact subset $[K]_{\mathcal{H}}$ of
$$[\Lambda_\Omega(\Gamma)]_{\mathcal{H}} - \{[x]_{\mathcal{H}}\},$$
where K is the (compact) preimage of $[K]_{\mathcal{H}}$ in $\Lambda_\Omega(\Gamma) - \{x\}$.

After taking a subsequence, $\gamma_n K$ must converge to a compact subset of $E_+ \cap \Lambda_\Omega(\Gamma)$, which does not intersect x'. In fact, part (3) of Proposition 5.15 implies that $\gamma_n K$ converges to a compact subset of $\Lambda_\Omega(\Gamma) - \overline{F'}$, where $F' = F_\Omega(x')$. So there is a fixed compact

$$K' \subset \Lambda_\Omega(\Gamma) - \overline{F'}$$

so that for sufficiently large n, $\gamma_n K \subset K'$. Then Lemma 8.16 implies that

$$h_n \gamma_n [K]_{\mathcal{H}} \subseteq h_n [K']_{\mathcal{H}}$$

sub-converges to $[\Lambda_\Omega(H_i)]_{\mathcal{H}}$. But on the other hand,

$$[(h_n \gamma_n) x_n]_{\mathcal{H}} \in C$$

sub-converges to some $b \neq [\Lambda_\Omega(H_i)]_{\mathcal{H}}$, so $h_n \gamma_n$ gives us the sequence of group elements we need. □

References

[BDL15] Samuel Ballas, Jeffrey Danciger, and Gye-Seon Lee. Convex projective structures on non-hyperbolic three-manifolds. *Geometry & Topology*, 22, 08 2015.

[Ben60] Jean-Paul Benzécri. Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. France*, 88:229–332, 1960.

[Ben00] Yves Benoist. Automorphismes des cônes convexes. *Invent. Math.*, 141(1):149–193, 2000.

[Ben03] Yves Benoist. Convexes hyperboliques et fonctions quasisymétriques. *Publ. Math. Inst. Hautes Études Sci.*, (97):181–237, 2003.

[Ben06a] Yves Benoist. Convexes divisibles. IV. Structure du bord en dimension 3. *Invent. Math.*, 164(2):249–278, 2006.

[Ben06b] Yves Benoist. Convexes hyperboliques et quasiisométries. *Geom. Dedicata*, 122:109–134, 2006.

[Ben08] Yves Benoist. A survey on divisible convex sets. In *Geometry, analysis and topology of discrete groups*, volume 6 of *Adv. Lect. Math. (ALM)*, pages 1–18. Int. Press, Somerville, MA, 2008.

[BK553] Chapter iv: Projective metrics. volume 3 of *Pure and Applied Mathematics*, pages 105–173. Elsevier, 1953.

[Bob20] Martin D. Bobb. Codimension-1 Simplices in Divisible Convex Domains. *arXiv e-prints*, page arXiv:2001.11096, January 2020.

[Bow98] Brian H. Bowditch. A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.*, 11(3):643–667, 1998.

[Bow99] B. H. Bowditch. Convergence groups and configuration spaces. In *Geometric group theory down under (Canberra, 1996)*, pages 23–54. de Gruyter, Berlin, 1999.

[Bow12] B. H. Bowditch. Relatively hyperbolic groups. *Internat. J. Algebra Comput.*, 22(3):1250016, 66, 2012.

[BPS19] Jairo Bochi, Rafael Potrie, and Andrés Sambarino. Anosov representations and dominated splittings. *J. Eur. Math. Soc. (JEMS)*, 21(11):3343–3414, 2019.

[BV23] Pierre-Louis Blayac and Gabriele Viaggi. Divisible convex sets with properly embedded cones. *arXiv e-prints*, page arXiv:2302.07177, February 2023.

[Cho10] Suhyoung Choi. The convex real projective orbifolds with radial or totally geodesic ends: The closedness and openness of deformations. *arXiv e-prints*, page arXiv:1011.1060, November 2010.

[CLM16] Suhyoung Choi, Gye-Seon Lee, and Ludovic Marquis. Convex projective generalized dehn filling. *Annales scientifiques de l'École normale supérieure*, 53, 11 2016.

[CLT15] D. Cooper, D. D. Long, and S. Tillmann. On convex projective manifolds and cusps. *Adv. Math.*, 277:181–251, 2015.

[CLT18] Daryl Cooper, Darren Long, and Stephan Tillmann. Deforming convex projective manifolds. *Geom. Topol.*, 22(3):1349–1404, 2018.

[CM14] Mickaël Crampon and Ludovic Marquis. Finitude géométrique en géométrie de Hilbert. *Ann. Inst. Fourier (Grenoble)*, 64(6):2299–2377, 2014.

[DGK17] Jeffrey Danciger, François Guéritaud, and Fanny Kassel. Convex cocompact actions in real projective geometry. *arXiv e-prints*, page arXiv:1704.08711, April 2017.

[DGK+21] Jeffrey Danciger, François Guéritaud, Fanny Kassel, Gye-Seon Lee, and Ludovic Marquis. Convex cocompactness for Coxeter groups. *arXiv e-prints*, page arXiv:2102.02757, February 2021.

[DS05] Cornelia Druţu and Mark Sapir. Tree-graded spaces and asymptotic cones of groups. *Topology*, 44(5):959–1058, 2005. With an appendix by Denis Osin and Mark Sapir.

[Fra91] Sidney Frankel. Applications of affine geometry to geometric function theory in several complex variables. I. Convergent rescalings and intrinsic quasi-isometric structure. In *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 183–208. Amer. Math. Soc., Providence, RI, 1991.

[Fre97] Eric M. Freden. Properties of convergence groups and spaces. *Conform. Geom. Dyn.*, 1:13–23 (electronic), 1997.

[GM87] F. W. Gehring and G. J. Martin. Discrete quasiconformal groups. I. *Proc. London Math. Soc. (3)*, 55(2):331–358, 1987.

[Gol88] William M. Goldman. Projective geometry on manifolds, 1988.

[GT87] M. Gromov and W. Thurston. Pinching constants for hyperbolic manifolds. *Invent. Math.*, 89(1):1–12, 1987.

[GW12] Olivier Guichard and Anna Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.*, 190(2):357–438, 2012.

[Isl19] Mitul Islam. Rank-One Hilbert Geometries. *arXiv e-prints*, page arXiv:1912.13013, December 2019.

[IZ19] Mitul Islam and Andrew Zimmer. Convex co-compact actions of relatively hyperbolic groups. *arXiv e-prints*, page arXiv:1910.08885, October 2019.

[IZ20] Mitul Islam and Andrew Zimmer. Convex co-compact representations of 3-manifold groups. *arXiv e-prints*, page arXiv:2009.05191, September 2020.

[IZ21] Mitul Islam and Andrew Zimmer. A flat torus theorem for convex co-compact actions of projective linear groups. *J. Lond. Math. Soc. (2)*, 103(2):470–489, 2021.

[IZ22] Mitul Islam and Andrew Zimmer. The structure of relatively hyperbolic groups in convex real projective geometry. *arXiv e-prints*, page arXiv:2203.16596, March 2022.

[Kap07] Michael Kapovich. Convex projective structures on Gromov-Thurston manifolds. *Geom. Topol.*, 11:1777–1830, 2007.

[KL18] Michael Kapovich and Bernhard Leeb. Relativizing characterizations of Anosov subgroups, I. *arXiv e-prints*, page arXiv:1807.00160, June 2018.

[KLP17] Michael Kapovich, Bernhard Leeb, and Joan Porti. Anosov subgroups: dynamical and geometric characterizations. *Eur. J. Math.*, 3(4):808–898, 2017.

[KLP18] Michael Kapovich, Bernhard Leeb, and Joan Porti. Dynamics on flag manifolds: domains of proper discontinuity and cocompactness. *Geom. Topol.*, 22(1):157–234, 2018.

[KS58] Paul Kelly and Ernst Straus. Curvature in Hilbert geometries. *Pacific J. Math.*, 8:119–125, 1958.

[Lab06] François Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.

[Mar14] Ludovic Marquis. Around groups in Hilbert geometry. In *Handbook of Hilbert geometry*, volume 22 of *IRMA Lect. Math. Theor. Phys.*, pages 207–261. Eur. Math. Soc., Zürich, 2014.

[Sul79] Dennis Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.

[Sul85] Dennis Sullivan. Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups. *Acta Math.*, 155(3-4):243–260, 1985.

[Tuk98] Pekka Tukia. Conical limit points and uniform convergence groups. *J. Reine Angew. Math.*, 501:71–98, 1998.

[Vin71] È. B. Vinberg. Discrete linear groups that are generated by reflections. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:1072–1112, 1971.

[Wei22] Theodore Weisman. An extended definition of Anosov representation for relatively hyperbolic groups. Preprint, arXiv:2205.07183, 2022. [math.GR].

[Yam04] Asli Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

[Zhu21] Feng Zhu. Relatively dominated representations. *Ann. Inst. Fourier (Grenoble)*, 71(5):2169–2235, 2021.

[Zim20] Andrew Zimmer. A higher rank rigidity theorem for convex real projective manifolds. *arXiv e-prints*, page arXiv:2001.05584, January 2020.

[Zim21] Andrew Zimmer. Projective Anosov representations, convex cocompact actions, and rigidity. *J. Differential Geom.*, 119(3):513–586, 2021.

[ZZ22] Feng Zhu and Andrew Zimmer. Relatively Anosov representations via flows I: theory. *arXiv e-prints*, page arXiv:2207.14737, July 2022.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Email address: tjwei@umich.edu