# Topological stability for (relatively) hyperbolic boundary actions

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Joint work and joint work in progress
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Basic stability question: if a group $\Gamma$ acts on a space $X$, how much does a "nearby" action look like the original one?

In this talk:

- $\Gamma$ is a (relatively) hyperbolic group
- $X$ is the Gromov (Bowditch) boundary $\partial\Gamma$
- Action is the standard boundary action $\Gamma \to \mathrm{Homeo}(\partial\Gamma)$.

Example: $\Gamma = \pi_1 M$ for $M$ closed (finite volume) hyperbolic, $X = \partial\mathbb{H}^n$.

In this context: this question is relevant for Mostow rigidity, hyperbolic Dehn filling, (higher) Teichmuller theory...

Consider nearby actions in $\mathrm{Hom}(\Gamma, \mathrm{Homeo}(\partial\Gamma))$.

Assuming $\partial\Gamma$ has a $C^1$ structure:

**Theorem (Sullivan 1985, Kapovich-Kim-Lee 2021)**

*Let $\rho : \Gamma \to \mathrm{Homeo}(\partial\Gamma)$ be standard boundary action, and suppose $\Gamma$ acts by $C^1$ maps. Any action $\rho' : \Gamma \to \mathrm{Homeo}(\partial\Gamma)$ which is sufficiently close to $\rho$ in the $C^1$ topology to $\rho$ is conjugate to $\rho$: for any $\gamma \in \Gamma$,*

$$\rho'(\gamma) = \phi \circ \rho(\gamma) \circ \phi^{-1}$$

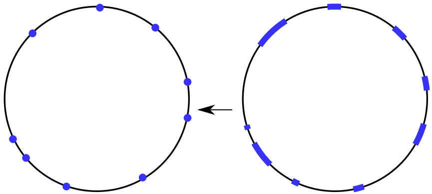
*for $\phi \in \mathrm{Homeo}(\partial\Gamma)$.*

Also a version where $\partial\Gamma$ does *not* have $C^1$ structure. But, this version also restricts to Lipschitz-close deformations.

What happens if we just perturb in the $C^0$ topology on $\mathrm{Homeo}(\partial\Gamma)$?

# Semi-conjugacy

If $\partial\Gamma = S^1$, can blow up points to intervals:



This can be done equivariantly with respect to $\Gamma$-action.

### Definition

$\Gamma$ acts on two topological spaces $X, Y$. A map $\phi : X \to Y$ is a *semi-conjugacy* if it is surjective and $\Gamma$-equivariant: for every $x \in X$,

$$\gamma \cdot \phi(x) = \phi(\gamma \cdot x)$$

The action of $\Gamma$ on $X$ "loses no information" from the action of $\Gamma$ on $Y$.

**Theorem (Mann-Manning-W, 2022)**

*Let $\rho : \Gamma \to \mathrm{Homeo}(\partial\Gamma)$ be standard boundary action. Any action $\rho' \in \mathrm{Hom}(\Gamma, \mathrm{Homeo}(\partial\Gamma))$ sufficiently close to $\rho$ is semi-conjugate to $\rho$.*

(also see: Gromov 1987)

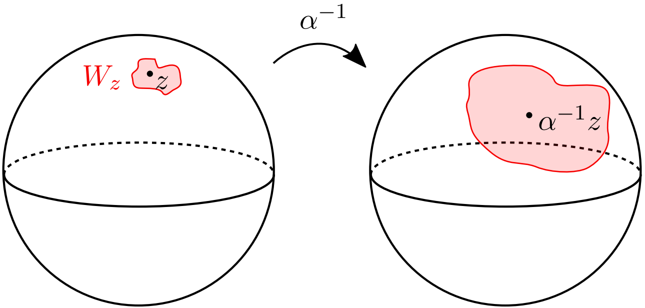
- Bowden-Mann, 2020: when $\Gamma = \pi_1 M$ for $M$ closed negatively curved Riemannian manifold
- Mann-Manning, 2021: when $\partial\Gamma$ homeomorphic to $S^n$

(Uses different proof strategy)

In progress: relative version (needs stronger hypotheses on perturbation)

Idea (Sullivan 1985): use expansion dynamics of action to find symbolic coding for points in $\partial\Gamma$.

**Given a point $x$ in $\partial\Gamma$, how can I find a (uniform) quasi-geodesic ray in $\Gamma$ limiting to $x$?**




Pick "expanding" neighborhood $W_z$ about each $z \in \partial\Gamma$.

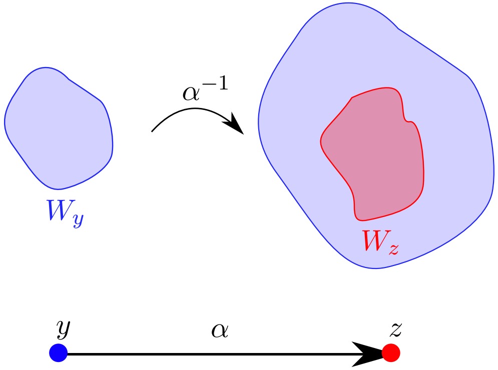
Unlike in Sullivan, "expansion" is measured topologically (visual metric is *not* essential).

Choose a finite cover $\{W_z\}_{z\in I}$ of $\partial\Gamma$ by expanding neighborhoods $W_z$.
Sets in cover = vertices of a directed graph $\mathcal{G}$, with edges labeled by expanding elements $\alpha \in \Gamma$.

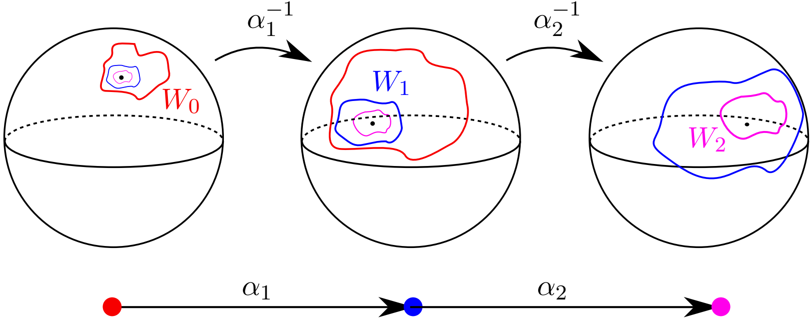
Rule: if there is an edge

$$y \xrightarrow{\alpha} z,$$

then $\alpha^{-1}W_y$ contains $\overline{W_z}$.



# Constructing codings




Path in $\mathcal{G}$ gives strictly nested sequence of subsets of $\partial\Gamma$, "coding" the unique point $x$ in intersection.

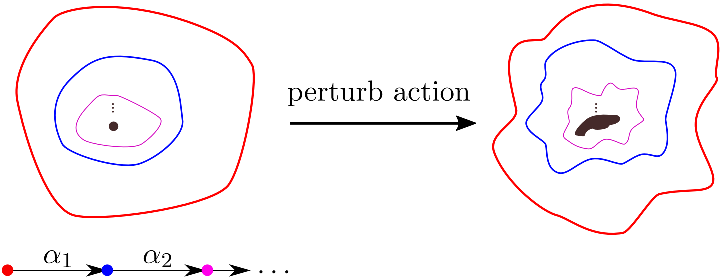
If the cover $\{W_z\}$ and graph $\mathcal{G}$ are constructed carefully:

- Every $x \in \partial\Gamma$ has a coding.
- The sequence $g_k = \alpha_1 \cdots \alpha_k$ is a uniform quasigeodesic with endpoint $x$.

# Constructing a semi-conjugacy

$$\left\{\begin{matrix}\text{Points in}\\ \partial\Gamma\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Infinite}\\ \text{paths in } \mathcal{G}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Closed}\\ \text{subsets of } \partial\Gamma\end{matrix}\right\}$$

$$\xleftarrow{\text{Semi-conjugacy } \phi}$$

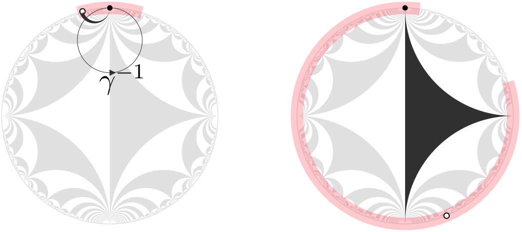



After perturbation, intersection may not be a singleton.

Verify: $\phi$ is well-defined, equivariant, surjective, continuous.

# The relative case

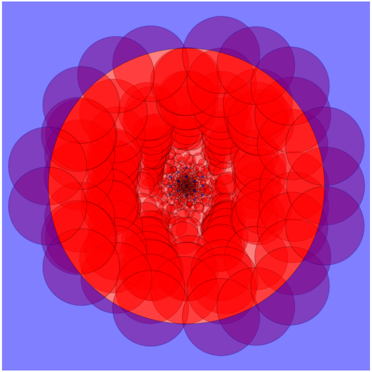
Problem: action is not "expanding" around a parabolic point $p$ in Bowditch boundary.

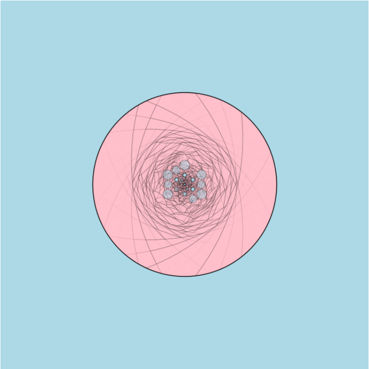



Still use an element of the parabolic subgroup to "expand" when coding points near $p$, but element to use depends on the point being coded.

$\mathcal{G}$ still has finitely many vertices, but each pair can have zero, one, or infinitely many directed edges between them.

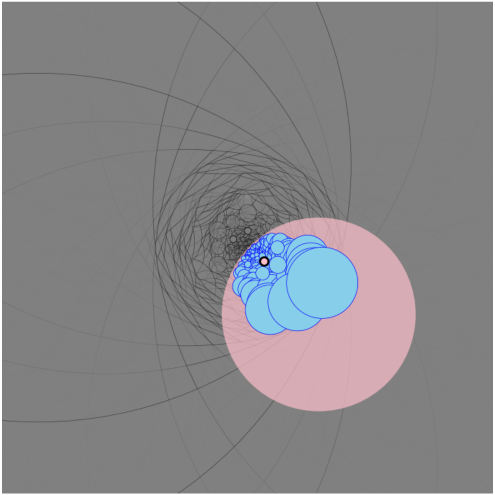
# Explicit constructions for the figure-eight knot group





This automaton has 1023 vertices and 628,771 directed edges (identifying multiple edges between the same pair of vertices).

# Explicit constructions for the figure-eight knot group



Vertex of automaton with 273 neighbors