Extended convergence dynamics and relative Anosov representations

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Goal: introduce a definition of relative Anosov representation and explain associated geometric structures.

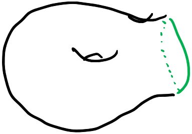
Convex cocompactness in hyperbolic space

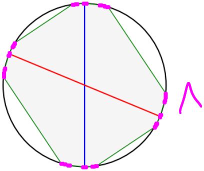
Def: $\Gamma \subset PO(d,1)$ discrete. ($PO(d,1) = \mathrm{Isom}(\mathbb{H}^d)$)

<u>Limit set</u> Λ_Γ = accumulation points of $\Gamma \cdot x$ in $\partial \mathbb{H}^d$ for $x \in \mathbb{H}^d$.

Γ is <u>convex cocompact</u> if Γ acts with compact quotient on $\mathrm{ConvHull}(\Lambda_\Gamma)$.

Ex: closed hyperbolic manifold

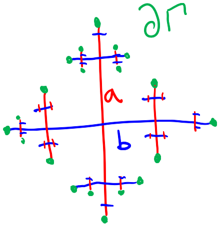




Dynamics of convex cocompact groups

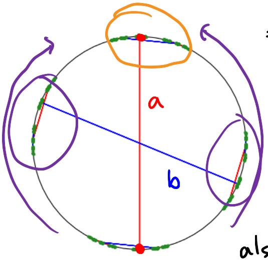
$\Gamma \subset PO(d,1)$ convex cocompact.

$\Longrightarrow$ Γ is word-hyperbolic, and Gromov boundary $\partial\Gamma$ embeds equivariantly into $\partial\mathbb{H}^d$ as Λ_Γ.



Cay(Γ)

Γ acts with north-south dynamics on $\partial\Gamma$



$\Rightarrow$ can play ping-pong on $\Lambda_\Gamma \subset \partial\mathbb{H}^2$

$\Rightarrow$ group elements close to a, b also generate a discrete free group.

[stable dynamics]

Stability:

Thm (Sullivan):

Let $\rho : \Gamma \to PO(d,1)$ be convex cocompact. Then an open neighborhood of ρ in $\mathrm{Hom}(\Gamma, PO(d,1))$ consists of convex cocompact representations.

Works even when Γ is <u>not</u> free.

("generalized ping-pong")

Definition

Let Γ be a hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *P_1-Anosov* if there exist equivariant embeddings

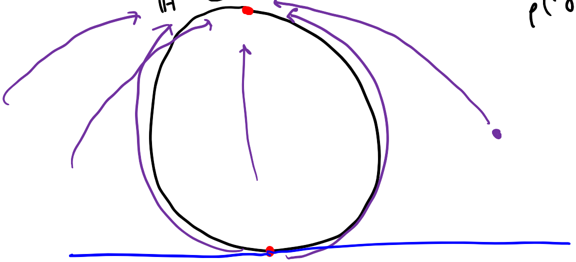
$$\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d), \qquad \xi^* : \partial\Gamma \to \underbrace{\mathbb{P}(\mathbb{R}^d)^*}_{\text{hyperplanes}}$$

which are *transverse*, and *preserve the dynamics of* Γ.

$\Gamma \cong \pi_1 S$ surface

$\rho : \Gamma \xrightarrow{\text{Fuchsian}} PO(2,1) \hookrightarrow PGL(3, \mathbb{R})$

$\mathbb{H}^2 \subset \mathbb{RP}^2$ round ball



$\gamma \in \Gamma$

$$\rho(\gamma) = \begin{bmatrix} \lambda & & \\ & 1 & \\ & & \lambda^{-1} \end{bmatrix}$$

north-south dynamics

extend to dense open subsets of $\mathbb{RP}^2$.

Thm (Labourie, Guichard-Wienhard):

Γ word-hyperbolic group. $\rho : \Gamma \to G$ Anosov representation.

An open neighborhood of ρ in $\mathrm{Hom}(\Gamma, G)$ consists of Anosov representations.

Q: what if Γ is not word-hyperbolic?

- what if Γ is relatively hyperbolic?
  - what discrete groups / geometric structures are there?
  - what kind of dynamics do we get?

Def: Discrete group $\Gamma \subset PO(d,1)$ is <u>geometrically finite</u> if Γ acts w/ finite covolume on an <u>ε-neighborhood</u> of $\mathrm{ConvHull}(\Lambda_\Gamma) \subseteq \mathbb{H}^d$.

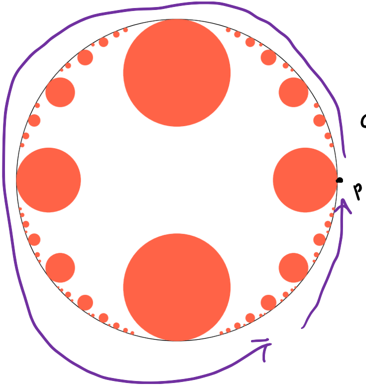
Ex: $\Gamma \subset PO(d,1)$ (nonuniform) lattice

Relative hyperbolicity:

$\Gamma \subset PO(d,1)$ geometrically finite.

Γ is hyperbolic relative to $\mathcal{H} = \{\text{cusp subgroups}\}$.

Limit set $\Lambda_\Gamma \subseteq \partial \mathbb{H}^d$ is identified w/ Bowditch boundary $\partial(\Gamma, \mathcal{H})$.



contains parabolic points

$\mathrm{Stab}_\Gamma(p)$ does not act w/ north-south dynamics.

Γ acts w/ convergence dynamics on $\partial \mathbb{H}^d$.

Def: A convex projective structure on a manifold M is a diffeomorphism $M \to \Omega/\Gamma$ for $\Omega \subset \mathbb{RP}^d$ properly convex and $\Gamma < PGL(d+1, \mathbb{R})$ discrete group preserving Ω.

Ex: any hyperbolic manifold (view $\mathbb{H}^d$ as a convex subset of $\mathbb{RP}^d$)

$\Gamma < PO(d,1)$ acts on $\mathbb{H}^d$, $\mathbb{H}^d/\Gamma$ is convex proj. manifold.

Thm: (Benoist) If $M = \Omega/\Gamma$ closed, and $\pi_1 M \cong \Gamma$ word-hyperbolic, $\Gamma \hookrightarrow PGL(d+1, \mathbb{R})$ is P_1-Anosov.

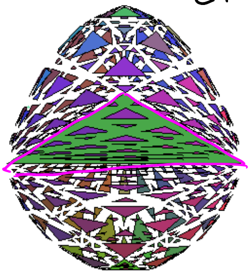
Thm (Danciger - Guéritaud - Kassel, Zimmer): Γ word-hyperbolic.

"Any Anosov representation can be associated to a convex proj. structure on a compact mfd in a mostly canonical way."

Convex projective manifolds with relatively hyperbolic fundamental group:

Benoist:

$C \mathbb{RP}^3$



$\mathbb{Z}^2 \curvearrowright$ triangle

$\downarrow$

incompressible torus

projective convex cocompact

$\Gamma \subset PGL(4, \mathbb{R})$

$\downarrow$

M

JSJ decomposition of M has hyperbolic pieces.

Other examples

- Ballas - Danciger - Lee
- Choi - Lee - Marquis
- Danciger - Guéritaud - Kassel - Lee - Marquis

Examples with cusps

- Ballas
- Ballas - Marquis
- Bobb

Definition

Let Γ be a hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is P_1-*Anosov* if there exist equivariant embeddings

$$\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d), \qquad \xi^* : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)^*$$

which are *transverse*, and *preserve the dynamics of* Γ.

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Definition (Kapovich - Leeb)

Let Γ be a relatively hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *relatively asymptotically embedded* if there exist equivariant embeddings

Bowditch bdry

$$\xi : \partial(\Gamma, \mathcal{P}) \to \mathbb{P}(\mathbb{R}^d), \quad \xi^* : \partial(\Gamma, \mathcal{P}) \to \mathbb{P}(\mathbb{R}^d)^*$$

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Definition

Let Γ be a relatively hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *relatively asymptotically embedded* if there exist equivariant embeddings

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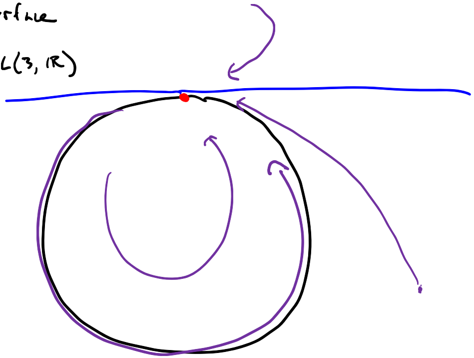
which are *transverse*, and *preserve the dynamics of* Γ.

$\Gamma \cong \pi_1 S$, S punctured surface

$\rho : \Gamma \to PO(2,1) \hookrightarrow PGL(3, \mathbb{R})$

$$\rho(\gamma) = \begin{bmatrix} 1 & \lambda & \lambda^2/2 \\ & 1 & \lambda \\ & & 1 \end{bmatrix}$$



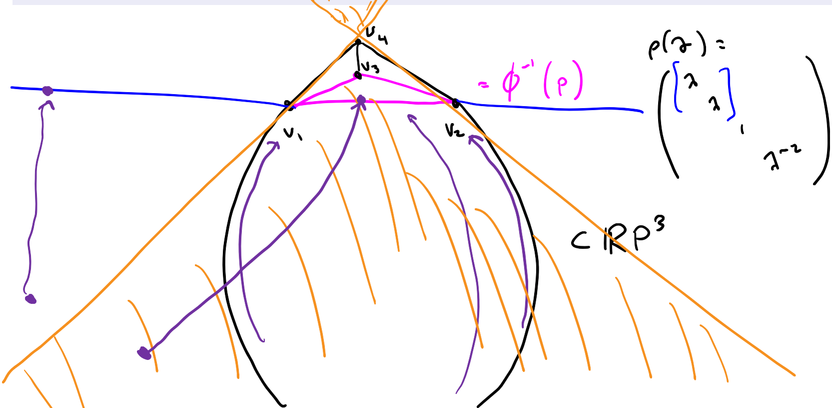
Definition (W.)

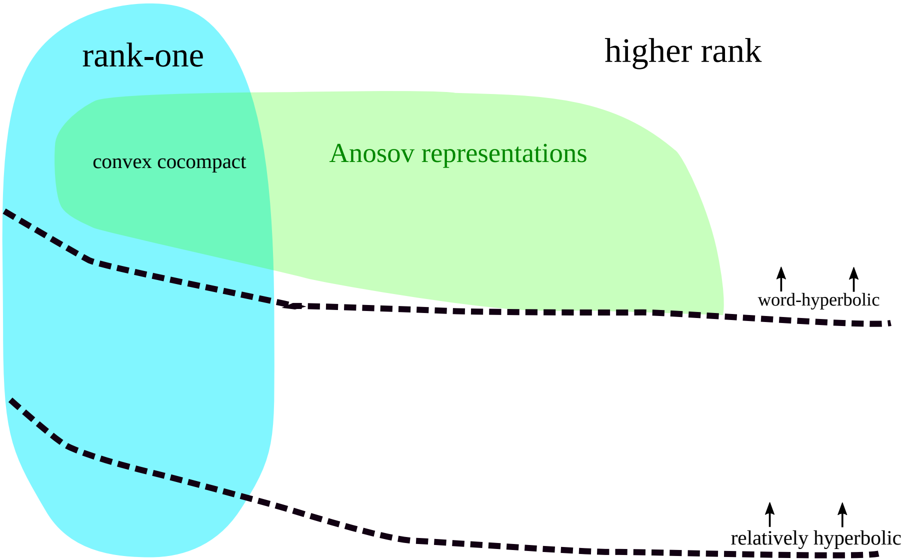
Let Γ be a relatively hyperbolic group. A representation $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ is *extended geometrically finite* if there exists a closed set $\Lambda \subset \mathcal{F}_{1,d-1}$ and a transverse equivariant *extension*

Bowditch boundary

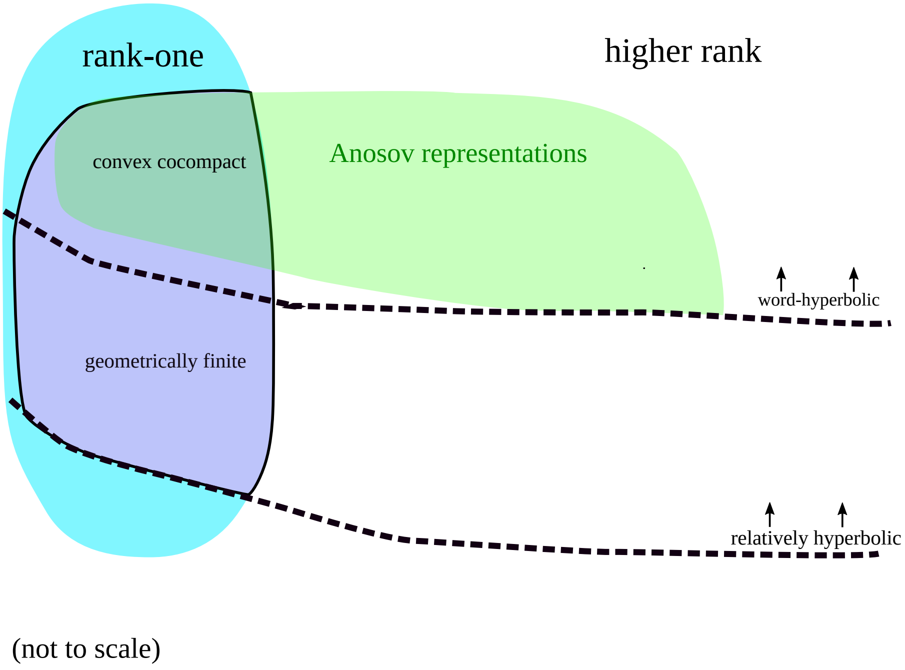
$$\phi : \Lambda \to \partial(\Gamma, \mathcal{P})$$

which *extends the convergence dynamics of* Γ.




rank-one
higher rank
convex cocompact
Anosov representations
word-hyperbolic
relatively hyperbolic

(not to scale)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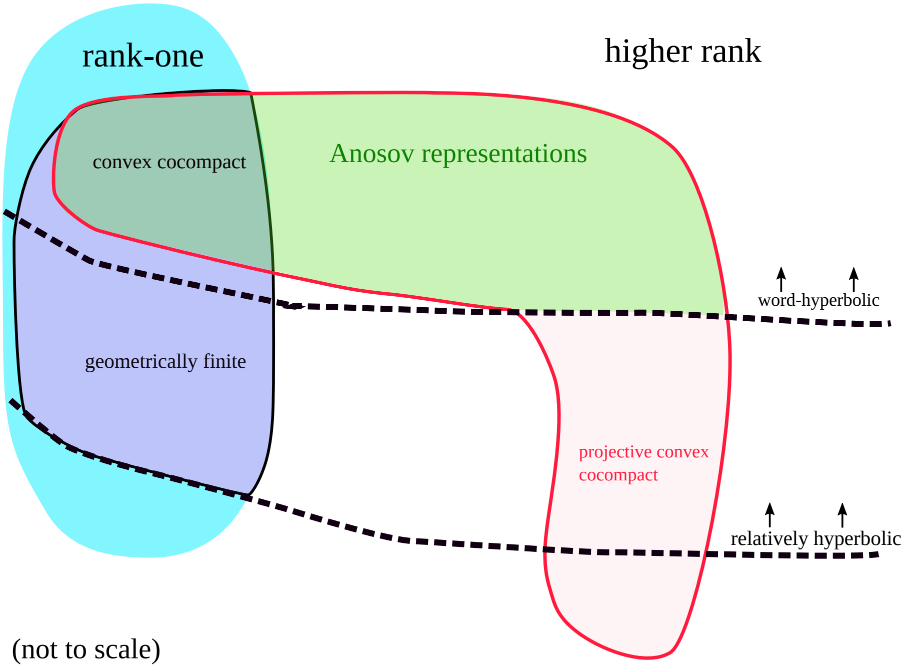
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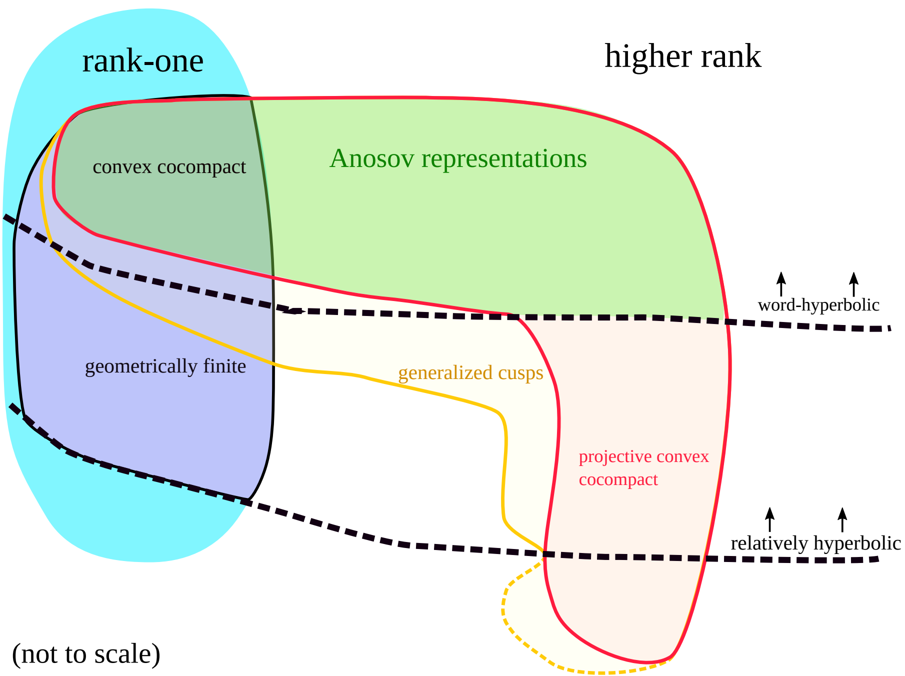
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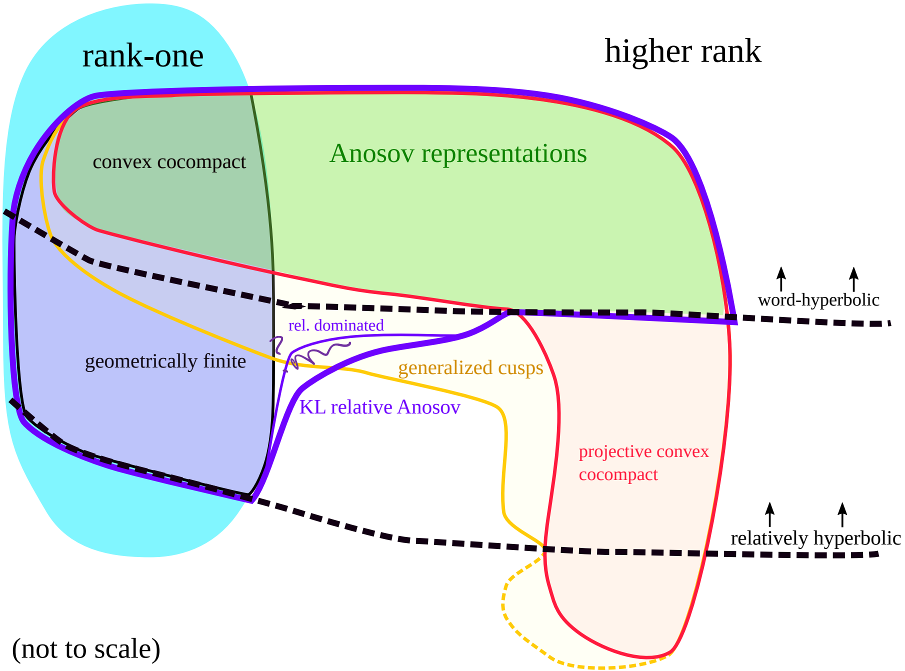
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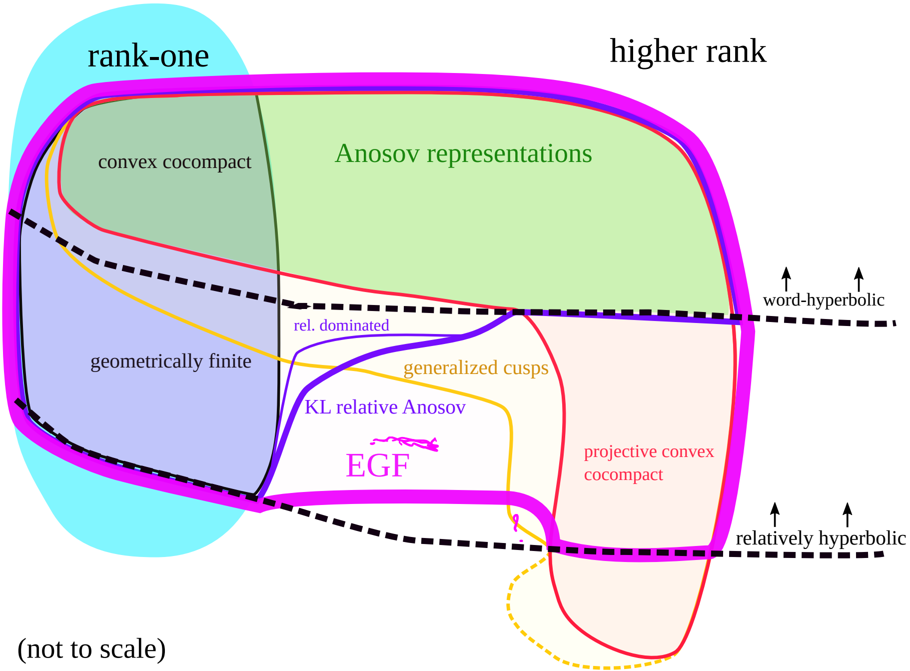
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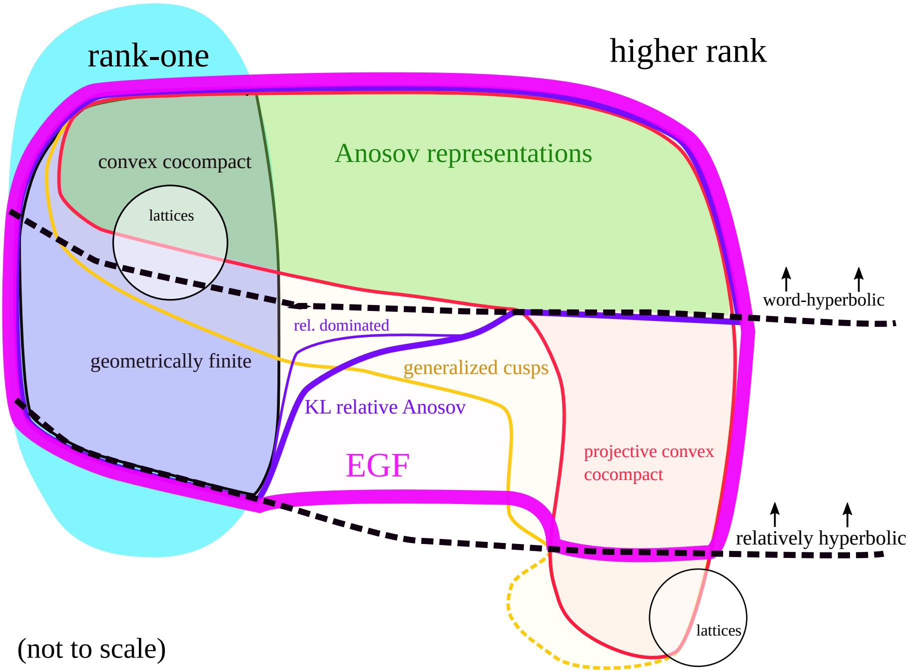
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(not to scale)

Γ is hyperbolic relative to subgroups $\mathcal{H} = \{H_i \subset \Gamma\}$.

Thm (W.):

Let $\rho : \Gamma \to G$ be an EGF representation, and let $W \subseteq \mathrm{Hom}(\Gamma, G)$ be a subspace which is peripherally stable at ρ. Then an open subset of W containing ρ consists of EGF representations.

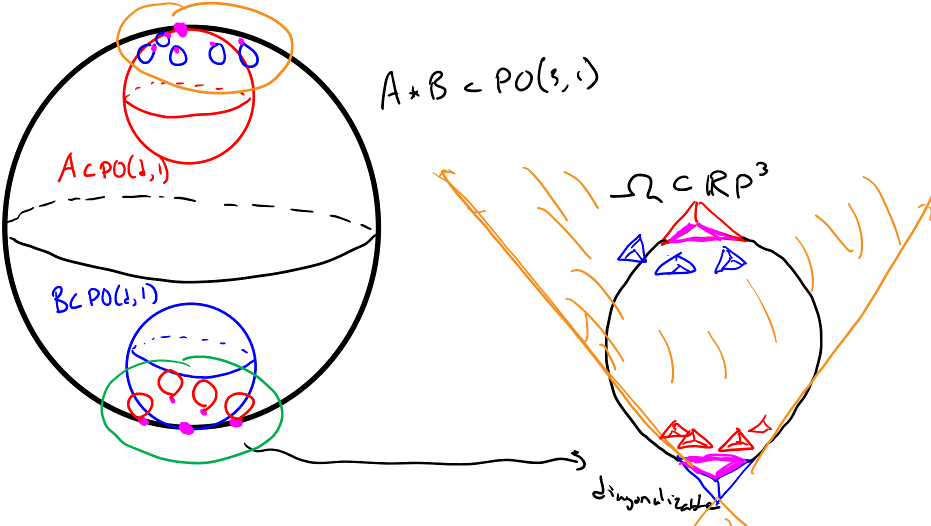
Cor: $\rho : \Gamma \to G$ EGF.

$$W = \{\rho' \in \mathrm{Hom}(\Gamma, G) : \rho'|_H \text{ conjugate to } \rho|_H \ \forall H \in \mathcal{H}\}$$

Open subset of W consists of EGF reps.

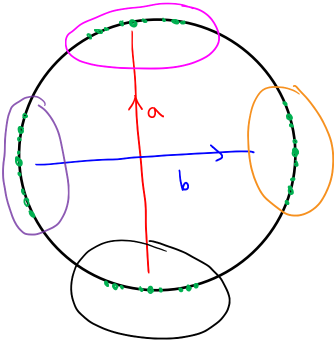
Peripheral Stability:

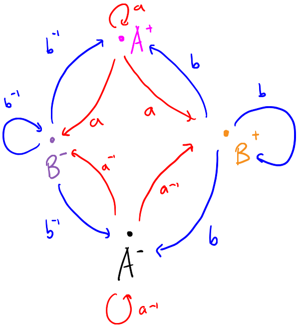
$\mathbb{H}^3 \subset \mathbb{RP}^3$:



Stability: proof idea

Cover limit set $\Lambda \subset F_{1,d}$ with finitely many open sets & construct finite directed graph with "ping pong" inclusions.





With parabolics: