ON FRACTIONAL SCHRÖDINGER EQUATIONS IN SOBOLEV SPACES

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Abstract. Let $\sigma \in (0,1)$ with $\sigma \neq \frac{1}{2}$. We investigate the fractional nonlinear Schrödinger equation in $\mathbb{R}^d$:

$$i\partial_t u + (-\Delta)^\sigma u + \mu |u|^{p-1} u = 0, \quad u(0) = u_0 \in H^s,$$

where $(-\Delta)^\sigma$ is the Fourier multiplier of symbol $|\xi|^{2\sigma}$, and $\mu = \pm 1$. This model has been introduced by Laskin in quantum physics [Las02]. We establish local well-posedness and ill-posedness in Sobolev spaces for power-type nonlinearities.

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1. Introduction

Let $\sigma \in (0,1)$ with $\sigma \neq \frac{1}{2}$. We consider the Cauchy problem for the fractional nonlinear Schrödinger equation

$$i\partial_t u + (-\Delta)^\sigma u + \mu |u|^{p-1} u = 0, \quad u(0) = u_0 \in H^s,$$  \text{ (NLS$_\sigma$)}

where $\mu = \pm 1$ depending on the focusing or defocusing case. The operator $(-\Delta)^\sigma$ is the so-called fractional laplacian, a Fourier multiplier of $|\xi|^{2\sigma}$. The fractional laplacian

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is the infinitesimal generator of some Levy processes [Ber96]. A rather extensive study of the potential theoretic aspects of this operator can be found in [Lan72].

The previous equation is a fundamental equation of fractional quantum mechanics, a generalization of the standard quantum mechanics extending the Feynman path integral to Levy processes [Las02].

The purpose of the present paper is to develop a general well-posedness and ill-posedness theory in Sobolev spaces. The one-dimensional case has been treated in [CHKL14] for cubic nonlinearities, i.e. $p = 3$, and $\sigma \in \left(\frac{1}{2}, 1\right)$. Here, we consider a higher-dimensional version and other types of nonlinear terms. We also include all $\sigma \in (0, 1)$ except $\sigma = \frac{1}{2}$; furthermore, contrary to [CHKL14] where the use of Bourgain spaces was crucial (since the main goal of their paper was to derive well-posedness theory on the flat torus), we rely only on standard Strichartz estimates and functional inequalities in $\mathbb{R}^d$. In the case of Hartree-type nonlinearities, the local well-posedness and blow-up have been investigated in [CHHO13].

In the present paper, we will not consider global aspects with large data. For that, we refer the reader to [GSWZ13] for a study of the energy-critical equation in the radial case, following the seminal work of Kenig and Merle [KM08, KM06]. As a consequence, we do not consider blow-up phenomena, an aspect we will treat in a forthcoming work.

We introduce two important exponents for our purposes:

$$s_c = \frac{d}{2} - \frac{2\sigma}{p - 1}$$

and

$$s_g = \frac{1 - \sigma}{2}.$$

Here, $s_c$ is the scaling-critical regularity exponent in the following sense: for $\lambda > 0$, the transformation

$$u(t, x) \mapsto \frac{1}{\lambda^{2\sigma/(p-1)}} u\left(\frac{t}{\lambda^{2\sigma}}, \frac{x}{\lambda}\right), \quad u_0(x) \mapsto \frac{1}{\lambda^{2\sigma/(p-1)}} u_0\left(\frac{x}{\lambda}\right)$$

keeps the equation invariant and one can expect local-wellposedness for $s \geq s_c$, since the scaling leaves the $\dot{H}^{s_c}$ norm invariant. On the other hand, $s_g$ is the critical regularity in the “pseudo”-Galilean invariance (see the proof of ill-posedness below).

Under the flow of the equation (NLS$_\sigma$), the following quantities are conserved:

$$M[u] = \int_{\mathbb{R}^d} |u(t, x)|^2 dx \quad \text{(mass)},$$

$$E[u] = \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{\mu}{p+1} |u(t, x)|^{p+1} dx \quad \text{(energy)}.$$
An important feature of the equation under study is a loss of derivatives for the Strichartz estimates as proved in [COX11]. Unless additional assumptions are met such as radiality as in [GW14], one has a loss of $d(1 - \sigma)$ derivatives in the dispersion (see (2.1)). This happens to be an issue in several arguments.

**Main results.** The goal of this paper is to show that $(\text{NLS}_{\sigma})$ is locally well-posed in $H^s$ for $s \geq \max(s_c, s_g, 0)$, and it is ill-posed in $H^s$ for $s \in (s_c, 0)$. We start with well-posedness results.

**Theorem 1.1** (Local well-posedness in subcritical cases). Let
\[
\begin{cases}
  s \geq s_g & \text{when } d = 1 \text{ and } 2 \leq p < 5, \\
  s > s_c & \text{when } d = 1 \text{ and } p \geq 5, \\
  s > s_c & \text{when } d \geq 2 \text{ and } p \geq 3.
\end{cases}
\]

Then, $(\text{NLS}_{\sigma})$ is locally well-posed in $H^s$.

**Theorem 1.2** (Local well-posedness in critical cases). Suppose that
\[
\begin{cases}
  p > 5 & \text{when } d = 1, \\
  p > 3 & \text{when } d \geq 2.
\end{cases}
\]

Then, $(\text{NLS}_{\sigma})$ is locally well-posed in $H^{s_c}$.

The proof of Theorem 1.2 is based on a new method, improving on estimates in [CKS+08]. This improvement, based on controlling the nonlinearity in a suitable space, is necessary due to the loss of derivatives in the Strichartz estimates.

As a by-product, we also prove small data scattering.

**Theorem 1.3** (Small data scattering). Suppose that
\[
\begin{cases}
  p > 5 & \text{when } d = 1, \\
  p > 3 & \text{when } d \geq 2.
\end{cases}
\]

Then, there exists $\delta > 0$ such that if $\|u_0\|_{H^{s_c}} < \delta$, then $u(t)$ scatters in $H^{s_c}$. Precisely, there exist $u_\pm \in H^{s_c}$ such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{it(-\Delta)^{\sigma}}u_\pm\|_{H^{s_c}} = 0.
\]

**Remark 1.4.** Contrary to the case $\sigma \neq \frac{1}{2}$, when $\sigma = \frac{1}{2}$, the fractional NLS does not have small data scattering. See [KLR13].

Finally, our last theorem is the ill-posedness result. Note that our result is not optimal, since one should expect ill-posedness in $H^s$ up to $s_g = \frac{1-\sigma}{2}$, which is nonnegative. We hope to come back to this issue in a forthcoming work.
Theorem 1.5 (Ill-posedness). Let $d = 1, 2$ or $3$ and $\sigma \in (\frac{d}{4}, 1)$. If $p$ is not an odd integer, we further assume that $p \geq k + 1$, where $k$ is an integer larger than $\frac{d}{2}$. Then, \((\text{NLS}_\sigma)\) is ill-posed in $H^s$ for $s \in (s_c, 0)$.

An interesting feature of the previous ill-posedness result is the fact that, contrary to the standard NLS equation ($\sigma = 1$) there is no exact Galilean invariance. However, one can introduce a new “pseudo-Galilean invariance” which is enough to our purposes. More precisely, for $v \in \mathbb{R}^d$, we define the transformation

$$
G_v u(t, x) = e^{-iv \cdot x} e^{it|v|^{2\sigma}} u(t, x - 2t|v|^{2(\sigma-1)}v).
$$

Note that when $\sigma = 1$, $G_v$ is simply a Galilean transformation, and that NLS is invariant under this transformation, that is, if $u(t)$ solves NLS, so does $G_v u(t)$. However, when $\sigma \neq 1$, \((\text{NLS}_\sigma)\) is not exactly symmetric with respect to pseudo-Galilean transformations. This opens the construction of solitons for \((\text{NLS}_\sigma)\) which happen to be different from the ones constructed in the standard case $\sigma = 1$. Indeed, if we search for exact solutions of the type

$$
u(t, x) = e^{it(|v|^{2\sigma} - \omega^{2\sigma})} e^{-iv \cdot x} Q_\omega(x - 2t\sigma|v|^{2(\sigma-1)}v),
$$

then the profile $Q_\omega$ solves the pseudo-differential equation

$$
P_v Q_\omega + \omega^{2\sigma} Q_\omega - |Q_\omega|^{p-1} Q_\omega = 0,
$$

where

$$
P_v = e^{iv \cdot x} (-\Delta)^\sigma e^{-iv \cdot x} - |v|^{2\sigma} - 2i\sigma|v|^{2\sigma-2}v \cdot \nabla,
$$

i.e., $P_v$ is a Fourier multiplier $\widehat{P_v f}(\xi) = p_v(\xi) \hat{f}(\xi)$, with symbol

$$
p_v(\xi) = |\xi - v|^{2\sigma} - |v|^{2\sigma} + 2\sigma|v|^{2\sigma-2}v \cdot \xi.
$$

We plan to come back to this issue in future works.

2. Strichartz Estimates

In this section, we review Strichartz estimates for the linear fractional Schrödinger operators. We say that $(q, r)$ is admissible if

$$
\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leq q, r \leq \infty, \quad (q, r, d) \neq (2, \infty, 2).
$$

We define the Strichartz norm by

$$
\|u\|_{S^r_{q, \tau}(I)} := \|\nabla^{-d(1-\sigma)(\frac{1}{2}-\frac{1}{r})} u\|_{L^q_{t \tau} W^r_x},
$$

where $I = [0, T)$. Let $\psi : \mathbb{R}^d \to [0, 1]$ be a compactly supported smooth function such that $\sum_{N \in 2^\mathbb{Z}} \psi_N = 1$, where $\psi_N(\xi) = \psi(\frac{\xi}{N})$. For dyadic $N \in 2^\mathbb{Z}$, let $P_N$ be a
Littlewood-Paley projection, that is, $\hat{P_Nf}(\xi) = \psi(\frac{\xi}{N})\hat{f}(\xi)$. Then, we define a slightly stronger Strichartz norm by

$$\|u\|_{S^s_{q,r}(I)} := \left( \sum_{N \in 2^\mathbb{Z}} \|P_N(\nabla^{-d(1-\sigma)(\frac{1}{2} - \frac{1}{p})} u)\|_{L_t^{2}L_x^{q,r}}^2 \right)^{1/2}.$$  

**Proposition 2.1** (Strichartz estimates [COX11]). For an admissible pair $(q,r)$, we have

$$\|e^{it(-\Delta)^\sigma}u_0\|_{S^s_{q,r}(I)}, \|e^{it(-\Delta)^\sigma}u_0\|_{S^s_{q,r}(I)} \lesssim \|u_0\|_{H^s},$$

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} F(s) \, ds \right\|_{S^s_{q,r}(I)} \lesssim \|F\|_{L_{t}^1 L_{x}^{q,r}};$$

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} F(s) \, ds \right\|_{S^s_{q,r}(I)} \lesssim \|F\|_{L_{t}^1 L_{x}^{q,r}}.$$  

**Sketch of Proof.** By the standard stationary phase estimate, one can show that

$$\|e^{it(-\Delta)^\sigma} P t \|_{L^1 \to L^\infty} \lesssim |t|^{-\frac{d}{2}},$$

and by scaling,

$$\|e^{it(-\Delta)^\sigma} P N \|_{L^1 \to L^\infty} \lesssim N^{d(1-\sigma)}|t|^{-\frac{d}{2}}. \quad (2.1)$$

Then, it follows from the argument of Keel-Tao [KT98] that for any $I \subset \mathbb{R}$,

$$\|e^{it(-\Delta)^\sigma} P N(\nabla^{-d(1-\sigma)(\frac{1}{2} - \frac{1}{p})} u_0)\|_{L_{t}^{q}L_{x}^{q,r}} \lesssim \|N u_0\|_{H^s},$$

$$\left\| \int_0^t e^{i(t-s)(-\Delta)^\sigma} P N(\nabla^{-d(1-\sigma)(\frac{1}{2} - \frac{1}{p})} F(s)) \, ds \right\|_{L_{t}^{q}L_{x}^{q,r}} \lesssim \|N F\|_{L_{t}^{q}L_{x}^{q,r}}.$$  

Squaring the above inequalities and summing them over all dyadic numbers in $2^\mathbb{Z}$, we prove Strichartz estimates. \qed

The loss of derivatives is due to the Knapp phenomenon (see [GW14]). However, in the radial case, one can overcome this loss as proved in [GW14], restricting then the admissible powers of the fractional laplacian. Indeed, in [GW14], this is proved that one has optimal Strichartz estimates if $\sigma \in (d/(2d-1), 1)$. In particular, the number $d/(2d-1)$ is larger than 1/2 and there is a gap between the Strichartz estimates for the wave operator $\sigma = 1/2$ and the one occuring for higher powers. This issue suggests that a new phenomenon might occur for this range of powers.

### 3. Local Well-posedness

We establish local well-posedness of the fractional NLS by the standard contraction mapping argument based on Strichartz estimates. Due to loss of regularity in Strichartz estimates, our proof relies on the $L^\infty_x$ bounds (see Lemma 3.2 and 3.3).
3.1. **Subcritical cases.** First, we consider the case that \( d = 1 \) and \( 2 \leq p < 5 \). In this case, the equation is scaling-subcritical in \( H^s \) for \( s > s_g \), since \( s_g > s_c \). We remark that in the proof, we control the \( L^4_t L^\infty_x \) norm simply by Strichartz estimates (see (3.1) and (3.2)).

**Proof of Theorem 1.1 when \( d = 1 \) and \( 2 \leq p < 5 \).** We define

\[
\Phi_{u_0}(u) := e^{it(-\Delta)^{s}} u_0 + i \mu \int_{0}^{t} e^{i(t-s)(-\Delta)^{s}} (|u|^{p-1}u)(s)ds.
\]

Let

\[
\|u\|_{X^s} := \|u\|_{L^4_t H^s \cap L^\infty_t L^\infty_x},
\]

where \( I = [0, T) \). Then, applying the 1d Strichartz estimates

\[
\|e^{it(-\Delta)^{s}} u_0\|_{L^4_t L^\infty_x} \lesssim \|u_0\|_{H^{s_g}}, \tag{3.1}
\]

\[
\|e^{it(-\Delta)^{s}} u_0\|_{L^\infty_t H^s_x} \lesssim \|u_0\|_{H^s},
\]

\[
\left\| \int_{0}^{t} e^{i(t-s)(-\Delta)^{s}} F(s)ds \right\|_{L^4_t L^\infty_x} \lesssim \|F\|_{L^1_t L^\infty_x H^{s_g}_x}, \tag{3.2}
\]

\[
\left\| \int_{0}^{t} e^{i(t-s)(-\Delta)^{s}} F(s)ds \right\|_{L^\infty_t H^s_x} \lesssim \|F\|_{L^1_t H^{s}_x},
\]

we get

\[
\|\Phi_{u_0}(u)\|_{X^s} \lesssim \|u_0\|_{H^s} + \|u\|^{p-1} \|u\|_{L^1_t H^s_x}.
\]

By the fractional chain rule

\[
\|\nabla|^{s} F(u)\|_{L^q} \lesssim \|F'(u)\|_{L^p} \|\nabla|^{s} u\|_{L^p}, \tag{3.3}
\]

where \( s > 0 \) and \( \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} \), and Hölder inequality, we obtain

\[
\|u|^{p-1}u\|_{L^1_t H^s_x} \lesssim \|u|^{p-1}\|_{L^q_x} \|u\|_{H^s_x} \lesssim T^{\frac{q-2}{q-1}} \|u\|^{p-1}_{L^1_t L^q_x} \|u\|_{L^\infty_t H^s_x}.
\]

For the fractional chain rule (3.3), we refer [CW91], for example. We remark that one can choose \( p_1 = \infty \) in (3.3). Indeed, this can be proved by a little modification of the last step in the proof of Proposition 3.1 in [CW91]. Thus, we have

\[
\|\Phi_{u_0}(u)\|_{X^s} \lesssim \|u_0\|_{H^s} + T^{\frac{q-2}{q-1}} \|u\|_{X^s}^p.
\]

Similarly, by Strichartz estimates,

\[
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X^s} \lesssim \|u|^{p-1}u - |v|^{p-1}v\|_{L^1_t H^s_x}.
\]
Then, applying the fractional Leibniz rule and the fractional chain rule in [CW91], we get
\[
\|u|^{p-1}-v|^{p-1}\|_{L^p_tL^q_x} = \left\| \int_0^1 p|v+t(u-v)|^{p-1}(u-v)dt \right\|_{L^q_x} \\
\leq p \left\| \int_0^1 |v+t(u-v)|^{p-1}(u-v)dt \right\|_{L^q_x} \\
\leq \int_0^1 \|v+t(u-v)\|^{p-1}_{L^q_x} \|u-v\|_{L^p_t} dt \\
+ \|v+t(u-v)\|^{p-1}_{L^q_x} \|u-v\|_{L^p_t} dt \\
\leq (\|u|^{p-1}_{L^p_tL^q_x} + \|v|^{p-1}_{L^p_tL^q_x}) \|u-v\|_{L^p_t} + (\|u|^{p-2}_{L^p_tL^q_x} + \|v|^{p-2}_{L^p_tL^q_x}) \|u-v\|_{L^q_x}.
\]

Thus, it follows that
\[
\| \Phi_{u_0}(u) - \Phi_{u_0}(v) \|_{X^s} \leq T^{\frac{p-1}{2}} \left\{ (\|u|^{p-1}_{L^{p_tL^q}} + \|v|^{p-1}_{L^{p_tL^q}}) \|u-v\|_{L^p_tL^q} + (\|u|^{p-2}_{L^{p_tL^q}} + \|v|^{p-2}_{L^{p_tL^q}}) \|u-v\|_{L^q_x} \right\} \\
\leq T^{\frac{p-1}{2}} (\|u\|^{p-1}_{X^s} + \|v\|^{p-1}_{X^s}) \|u-v\|_{X^s}.
\]
Choosing sufficiently small \( T > 0 \), we conclude that \( \Phi_{u_0} \) is a contraction on a ball
\[
B := \{ u : \|u\|_{X^s} \leq 2\|u_0\|_{H^s} \}
\]
equipped with the norm \( \| \cdot \|_{X^s} \). \qed

Next, we will prove Theorem 1.1 when \( d = 1 \) and \( p \geq 5 \), or \( d \geq 2 \) and \( p \geq 3 \). In this case, we do not have a good control on the \( L^\infty_x \) norm from Strichartz estimates. Instead, we make use of Sobolev embedding.

**Lemma 3.1** \((L^{p-1}_{t\in I}L^\infty_x) \text{ bound})\). Suppose that \( d = 1 \) and \( p \geq 5 \), or \( d \geq 2 \) and \( p \geq 3 \). Let \( s > s_c \). Then, we have
\[
\|u\|_{L^{p-1}_{t\in I}L^\infty_x} \leq T^{0+}\|u\|_{S^s_{q_0,r_0}(I)}, \tag{3.4}
\]
where \((q_0,r_0) = ((p-1)^+, \left( \frac{2d(p-1)}{d(p-1)-4} \right)^-)\) is an admissible pair. Here, we denote by \( c^+ \) a number larger than \( c \) but arbitrarily close to \( c \), and similarly for \( c^- \).
Proof. We observe that

$$\frac{1}{r_0} - \frac{s - d(1 - \sigma)(\frac{1}{2} - \frac{1}{r_0})}{d} < 0.$$  

Thus, by Sobolev inequality,

$$\|u\|_{L_{t\in I}^p L_x^q} \leq T_0^+ \|u\|_{L_{t\in I}^p L_x^q} \leq \|\nabla|^{-d(1-\sigma)(\frac{1}{2} - \frac{1}{r_0})}u\|_{L_{t\in I}^p W_x^{s_2,r_0}} = \|u\|_{S_{s_0,r_0}^s(I)}.$$

We also employ a standard persistence of regularity argument.

**Lemma 3.2** (Persistence of regularity). Let $1 < q \leq \infty$, $1 < r < \infty$ and $s_1 \geq s_2$. Then, $B = \{u : \|u\|_{L_{t\in I}^p W_x^{s_1,r}} \leq R\}$, equipped with the norm $\|\cdot\|_{L_{t\in I}^p W_x^{s_2,r}}$, is a complete metric space.

**Proof.** We recall:

**Theorem 3.3** (Theorem 1.2.5 in [Caz03]). Consider two Banach spaces $X \hookrightarrow Y$ and $1 < p, q \leq \infty$. Let $(f_n)_{n \geq 0}$ be a bounded sequence in $L^q(I,Y)$ and let $f : I \to Y$ be such that $f_n(t) \to f(t)$ in $Y$ as $n \to \infty$, for a.e. $t \in I$. If $(f_n)_{n \geq 0}$ is bounded in $L^p(I,X)$ and if $X$ is reflexive, then $f \in L^p(I;X)$ and $\|f\|_{L^p(I;X)} \leq \liminf_{n \to \infty} \|f_n\|_{L^p(I;X)}$.

Suppose that $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $B$. Then, $f_n$ converges to $f$ in $L_{t\in I}^q W_x^{s_2,r}$. Moreover, it follows from Theorem 1.2.5 in [Caz03] that

$$\|f\|_{L_{t\in I}^q W_x^{s_2,r}} \leq \liminf_{n \to \infty} \|f_n\|_{L_{t\in I}^q W_x^{s_1,r}} \leq R,$$

and thus $f \in B$. Therefore, we conclude that $B$ is complete. □

**Proof of Theorem 1.1 when $d = 1$ and $p \geq 5$, or $d \geq 2$ and $p \geq 3$.** Define the map $\Phi_{u_0}$ as above, and let

$$X^\alpha := L_{t\in I}^\infty H_x^\alpha \cap S_{q_0,r_0}^\alpha(I),$$

where $(q_0, r_0)$ is an admissible pair in Lemma 3.2. Then, by Strichartz estimates, the fractional chain rule and (3.4), we get

$$\|\Phi_{u_0}(u)\|_{X^s} \leq \|u_0\|_{H^s} + \|u\|_{L_{t\in I}^p H_x^s} \leq \|u_0\|_{H^s} + \|u\|_{L_{t\in I}^p L_x^p} \|u\|_{L_{t\in I}^p W_x^{s,r_0}} \leq \|u_0\|_{H^s} + T_0^+ \|u\|_{S_{q_0,r_0}^s(I)} \|u\|_{L_{t\in I}^p H_x^s} \leq \|u_0\|_{H^s} + T_0^+ \|u\|_{X^s}.$$
and similarly
\[
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{X^0} \lesssim \|u|^{p-1}u - |v|^{p-1}v\|_{L_{te}^1 L_x^2}^{1/2} \\
\lesssim \|(|u|^{p-1} + |v|^{p-1})|u - v|\|_{L_{te}^1 L_x^2} \\
\lesssim \left(\|u\|_{L_{te}^1 L_x^\infty}^{p-1} + \|v\|_{L_{te}^1 L_x^\infty}^{p-1}\right)\|u - v\|_{L_{te}^1 L_x^2} \\
\lesssim T^{0+}\left(\|u\|_{S^1_{\theta,0} \cap \{t\}}^{p-1} + \|v\|_{S^1_{\theta,0} \cap \{t\}}^{p-1}\right)\|u\|_{L_{te}^\infty L_x^2} \\
\lesssim T^{0+}\left(\|u\|_{X^0}^{p-1} + \|v\|_{X^0}^{p-1}\right)\|u - v\|_{X^0}.
\]

Thus, for sufficiently small \( T > 0 \), \( \Phi_{u_0} \) is contractive on a ball
\[
B := \{u : \|u\|_{X^0} \leq 2\|u_0\|_{H^s}\}
\]
equipped with the norm \( \|\cdot\|_{X^0} \), which is complete by Lemma 3.2. \( \square \)

Remark 3.4. The standard persistence of regularity argument allows us to avoid derivatives in (3.5). Indeed, for \( u \in B \), \( \|\nabla^s u\|_{L_{te}^1 L_x^\infty} \) is not necessarily bounded.

3.2. Scaling-critical cases. In the scaling-critical case, we use the following lemma, which plays the same role as (3.4). We note that the norms in the lemma are defined via the Littlewood-Paley projection in order to overcome the failure of the Sobolev embedding \( W^{s,p} \hookrightarrow L^q \), \( 1/q = 1/p - \frac{s}{d} \), when \( q = \infty \). Lemma 3.3 generalizes [CKS+08, Lemma 3.1].

Lemma 3.5 (Scaling-critical \( L_{te}^1 L_x^\infty \) bound).
\[
\|u\|_{L_{te}^1 L_x^\infty}^{p-1} \lesssim \begin{cases} 
\|u\|_{S^1_{\theta,0} \cap \{t\}}^{p-5} & \text{when } d = 1 \text{ and } p > 5, \\
\|u\|_{S^2_{\theta,0} \cap \{t\}}^{p-3} & \text{when } d = 2 \text{ and } p > 3, \\
\|u\|_{S^3_{\theta,0} \cap \{t\}}^{p-3} & \text{when } d \geq 3 \text{ and } p > 3.
\end{cases}
\]

Proof. We will prove the lemma only when \( d \geq 3 \). By interpolation \( \|f\|_{L_p^\theta} \leq \|f\|_{L_p^0} \|f\|_{L_p^1}^{1-\theta}, \frac{1}{p_0} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}, 0 < \theta < 1 \), it suffices to show the lemma for rational \( (p-1) = \frac{m}{n} > 2 \) with \( \gcd(m,n) = 1 \). First, we estimate
\[
A(t) = \left[ \sum_{N} \|P_N u(t)\|_{L_x^\infty} \right]^{m} \sim \sum_{N \gg \cdots \gg N_m} \prod_{i=1}^{m} \|P_N u(t)\|_{L_x^\infty}.
\]
Observe from Bernstein’s inequality that
\[
\|P_N u(t)\|_{L_x^\infty} \lesssim N^{-\frac{(p-1)}{p-1}} d_N, \quad (3.7)
\]
\[
\|P_N u(t)\|_{L_x^2} \lesssim N^{-\frac{2}{p-1}} d'_N, \quad (3.8)
\]
where
\[ d_N = \| P_N u(t) \|_{W^{\sigma, \frac{p}{p-1}, \frac{p}{2}}} \quad \text{and} \quad d'_N = \| P_N u(t) \|_{H^{\sigma, \frac{p}{2}}} \]  

As a consequence, we have
\[ \| P_N u(t) \|_{L^p} \lesssim \left( N^{-\frac{\sigma(p-3)}{p-1}} d_N \right)^{\theta} N^{\frac{2\sigma}{p-2}} \left( d'_N \right)^{1-\theta} = N^{-\frac{\sigma(p-3)}{p-1}(p-2)} (d_N)^{\theta} (d'_N)^{1-\theta}, \]  
where \( \theta = \frac{1}{p-2} \). Hence, applying (3.7) for \( i = 1, \cdots, n \) and (3.9) for \( i = n+1, \cdots, m \), we bound \( A(t) \) by
\[ \lesssim \sum_{N_1 \geq \cdots \geq N_m} \left( \prod_{i=1}^n N_i^{-\frac{\sigma(p-3)}{p-1}} \epsilon d_N \right) \left( \prod_{i=n+1}^m N_i^{-\frac{\sigma(p-3)}{p-1}(p-2)} \epsilon (d'_N \}^\theta \right). \]  
For an arbitrarily small \( \epsilon > 0 \), we let
\[ \tilde{d}_N = \sum_{N' \in 2^n} \min \left( \frac{N'}{N}, \frac{N'}{N} \right)^\epsilon d_N', \quad \tilde{d}'_N = \sum_{N' \in 2^n} \min \left( \frac{N'}{N}, \frac{N'}{N} \right)^\epsilon d'_N. \]  
Then, since \( d_N \leq \tilde{d}_N \) and \( d_N' \leq (N_i/N) \tilde{d}_N \), and similarly for primes, \( A(t) \) is bounded by
\[ \lesssim \sum_{N_1 \geq \cdots \geq N_m} \left( \prod_{i=1}^n N_i^{-\frac{\sigma(p-3)}{p-1}} \epsilon \tilde{d}_N \right) \left( \prod_{i=n+1}^m N_i^{-\frac{\sigma(p-3)}{p-1}(p-2)} \epsilon \tilde{d}'_N \right). \]  
Summing in \( N_m, N_{m-1}, \ldots, N_{n+1} \) and using \( m-n = (p-2)n \),
\[ A(t) \lesssim \sum_{N_1 \geq \cdots \geq N_n} \left( \prod_{i=1}^n N_i^{-\frac{\sigma(p-3)}{p-1}} \epsilon \tilde{d}_N \right) \times N_n^{-\frac{\sigma(p-3)(m-n)}{p-1}} \left( \frac{N_1}{N} \right)^{(m-n)\epsilon} \epsilon (d_N)^{(m-n)\theta} (d'_N)^{(m-n)(1-\theta)} \]  
and then summing in \( N_n, N_{n-1}, \ldots, N_1 \), we obtain that
\[ A(t) \lesssim \sum_{N_1} (\tilde{d}_N)^{n+(m-n)\theta} (d'_N)^{(m-n)(1-\theta)} = \sum_{N_1} (\tilde{d}_N)^{2n} (d'_N)^{m-2n}, \]  
which is, by Hölder inequality and Young’s inequality, bounded by
\[ \lesssim \| (\tilde{d}_N)^{2n} \|_{\ell^2_N} \| (d'_N)^{m-2n} \|_{\ell^2_N} = \| \tilde{d}_N \|_{\ell^2_N}^{2n} \| d'_N \|_{\ell^2_N}^{m-2n}, \]  
\[ \lesssim \| \tilde{d}_N \|_{\ell^2_N}^{2n} \| \tilde{d}_N \|_{\ell^2_N}^{m-2n} \lesssim \| d_N \|_{\ell^2_N}^{2n} \| d'_N \|_{\ell^2_N}^{m-2n} = \| d_N \|_{\ell^2_N}^{2n} \| d'_N \|_{\ell^2_N}^{(p-3)n}. \]
Finally, by the estimate for $A(t)$, we prove that
\[
\|u\|_{L_t^{p-1} L_x^2}^{p-1} \leq \int_I A(t)^{p-1} dt = \int_I A(t)^{\frac{m}{n}} dt
\]
\[
\leq \int_I \|d_N\|_{\ell_N^2}^2 \|d_N'\|_{\ell_N^2}^{p-3} dt \leq \|d_N\|_{L_t^{\infty} \ell_N^2}^2 \|d_N'\|_{L_t^{\infty} \ell_N^2}^{p-3}
\]
\[
\leq \|u\|_{\tilde{S}^{s_e, 2 d / m - 2} (I)}^2 \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)}^{p-2}.
\]

Proof of theorem 1.2. For simplicity, we assume that $d \geq 3$. Indeed, with little modifications, we can prove the theorem when $d = 1, 2$. We define $\Phi_{u_0}(u)$ as in the proof of Theorem 1.1. Then, by Strichartz estimates, the fractional chain rule and (3.6), we have
\[
\|\Phi_{u_0}(u)\|_{\tilde{S}^{s_e, 2 d / m} (I)} \leq \|e^{it(-\Delta)^s} u_0\|_{\tilde{S}^{s_e, 2 d / m} (I)} + c_0 \|u\|_{L_t^1 H^{s_e}}^{p-1} \|u\|_{L_t^1 \ell_N^2}^1
\]
\[
\leq \|e^{it(-\Delta)^s} u_0\|_{\tilde{S}^{s_e, 2 d / m} (I)} + c_1 \|u\|_{L_t^1 \ell_N^2} \|u\|_{L_t^\infty H^{s_e}}
\]
\[
\leq \|e^{it(-\Delta)^s} u_0\|_{\tilde{S}^{s_e, 2 d / m} (I)} + c \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)} \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)}^{p-2}.
\]

Similarly, one can show that
\[
\|\Phi_{u_0}(u)\|_{\tilde{S}^{s_e, 2 d / m} (I)} \leq c \|u_0\|_{H^{s_e}} + c \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)} \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)}^{p-2}
\]

and
\[
\|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{\tilde{S}^{s_e, 2 d / m} (I)} + \|\Phi_{u_0}(u) - \Phi_{u_0}(v)\|_{\tilde{S}^{s_e, 2 d / m} (I)}
\]
\[
\leq c_0 \|u\|_{L_t^1 \ell_N^2}^{p-1} \|v\|_{L_t^1 \ell_N^2} \|u - v\|_{L_t^\infty \ell_N^2}
\]
\[
\leq c \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)} \|v\|_{\tilde{S}^{s_e, 2 d / m} (I)} \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)} \|v\|_{\tilde{S}^{s_e, 2 d / m} (I)} \|u - v\|_{L_t^\infty \ell_N^2}
\]

Now we let $\delta = \delta(c, \|u_0\|_{H^{s_e}}) > 0$ be a sufficiently small number to be chosen later, and then we pick $T = T(u_0, \delta) > 0$ such that
\[
\|e^{it(-\Delta)^s} u_0\|_{\tilde{S}^{s_e, 2 d / m} (I)} \leq \delta,
\]

Define
\[
B = \left\{ u : \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)} \leq 2 \delta \text{ and } \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)} \leq 2c \|u_0\|_{H^{s_e}} \right\}
\]

equipped with the norm
\[
\|u\|_X := \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)} + \|u\|_{\tilde{S}^{s_e, 2 d / m} (I)}
\]
Then, for \( u \in B \), we have
\[
\| \Phi_{u_0}(u) \|_{\tilde{S}_{2,q,2d}^{sc}(I)} \leq \delta + c(2\delta)^2(2c\|u_0\|_{H^{sc}})^{p-2} \leq 2\delta,
\]
\[
\| \Phi_{u_0}(u) \|_{\tilde{S}_{2,q,2d}^{sc}(I)} \leq c\|u_0\|_{H^{sc}} + c(2\delta)^2(2c\|u_0\|_{H^{sc}})^{p-2} \leq 2c\|u_0\|_{H^{sc}}.
\]
Choosing sufficiently small \( \delta > 0 \), we prove that \( \Phi_{u_0} \) maps \( B \) to itself. Similarly, one can show
\[
\| \Phi_{u_0}(u) - \Phi_{u_0}(v) \|_X \leq \frac{1}{2}\| u - v \|_X.
\]
Therefore, it follows that \( \Phi_{u_0} \) is a contraction mapping in \( B \).

**Remark 3.6.** (i) In the proofs, the \( L^p_X \) norm bounds are crucial for the following reason. In Proposition 2.1, there is a loss of regularity except the trivial ones,
\[
\| e^{it(-\Delta)^\sigma} u_0 \|_{L^p_{t \in I} L^2_X} = \| u_0 \|_{L^2}
\]
and
\[
\left\| \int_0^t e^{it(-\Delta)^\sigma} F(s)ds \right\|_{L^p_{t \in I} L^2_X} \leq \| F \|_{L^p_{t \in I} L^2_X}.
\]
Hence, when we estimate the \( L^\infty_{t \in I} H^s_X \) norm of the integral term in \( \Phi_{u_0}(u) \), we are forced to use the trivial one
\[
\left\| \int_0^t e^{it(-\Delta)^\sigma} |u|^{p-1} u(s)ds \right\|_{L^\infty_{t \in I} H^s_X} \leq \| |u|^{p-1} u \|_{L^1_{t \in I} H^s_X}.
\]
Indeed, otherwise, we have a higher regularity norm on the right hand side. Then, we cannot close the contraction mapping argument. Moreover, if \( u_0 \in H^s \), there is no good bound for \( \| e^{it(-\Delta)^\sigma} u_0 \|_{L^q_{t \in I} W^{q,r}_X} \) except the trivial one \((q,r) = (\infty,2)\). Thus, we are forced to bound the right hand side of (3.10) by
\[
\| u \|_{L^p_{t \in I} L^\infty_X} \leq \| u \|_{L^p_{t \in I} L^\infty_X}.
\]
Therefore, we should have a good control on \( \| u \|_{L^p_{t \in I} L^\infty_X} \).
(ii) When \( p < 3 \), the \( L^p_{t \in I} L^\infty_X \) norm is scaling-supercritical. Thus, based on our method, the assumptions on \( p \) in Theorem 1.1 and 1.2 are optimal except \( p = 3 \) in the critical case.

### 4. Small Data Scattering

**Proof of Theorem 1.3.** For simplicity, we consider the case \( d \geq 3 \) only. It follows from the estimates in the proof of Theorem 1.2 that if \( \| u_0 \|_{H^s} \) is small enough, then
\[
\| u(t) \|_{L^p_{t \in I} L^\infty_X} + \| u(t) \|_{L^\infty_{t \in I} H^s_X} \leq \| u(t) \|_{\tilde{S}_{2,q,2d}^{sc}(\mathbb{R})} + \| u(t) \|_{\tilde{S}_{2,q,2d}^{sc}(\mathbb{R})} \lesssim \| u_0 \|_{H^{sc}} < \infty.
\]
By Strichartz estimates, the fractional chain rule and (3.6), we prove that
\[
\|e^{-iT_1(-\Delta)u(T_1)} - e^{-iT_2(-\Delta)u(T_2)}\|_{H^{s_c}}
\]
\[
= \left\| \int_{T_1}^{T_2} e^{-is(-\Delta)^\sigma(|u|^{p-1}u)(s)} ds \right\|_{H^{s_c}}
\]
\[
\leq \left\| u(t) \right\|_{L^\infty_{t \in [T_1, T_2], L^p_x}}^{p-1} \| u(t) \|_{L^\infty_{t \in [T_1, T_2], H^{s_c}}} \rightarrow 0
\]
as \(T_1, T_2 \rightarrow \pm \infty\). Thus, the limits
\[
u(t) = \lim_{t \rightarrow \pm \infty} e^{-it(-\Delta)^\sigma} u(t)
\]
exist in \(H^{s_c}\). Repeating the above estimates, we show that
\[
\|u(t) - e^{it(-\Delta)^\sigma} u(t)\|_{H^{s_c}} = \|e^{-it(-\Delta)^\sigma} u(t) - u(t)\|_{H^{s_c}} \rightarrow 0
\]
as \(t \rightarrow \pm \infty\). □

5. ILL-POSEDNESS

We will prove Theorem 1.5 following the strategy in [CCT03a]. Throughout this section, we assume that \(d = 1, 2\) or \(3\) and \(\frac{d}{4} < \sigma < 1\). If \(p\) is not an odd integer, we further assume that \(p \geq k + 1\), where \(k\) is the smallest integer greater than \(\frac{d}{2}\).

First, we construct an almost non-dispersive solution by small dispersion analysis.

Lemma 5.1 (Small dispersion analysis). Given a Schwartz function \(\phi_0\), let \(\phi^{(\nu)}(t, x)\) be the solution to the fractional NLS
\[
i\partial_t u + \nu^{2\sigma}(-\Delta)^\sigma u + \mu|u|^{p-1}u = 0, \quad u(0) = \phi_0,
\]
and \(\phi^{(0)}(t, x)\) be the solution to the ODE with no dispersion
\[
i\partial_t u + \mu|u|^{p-1}u = 0, \quad u(0) = \phi_0,
\]
that is,
\[
\phi^{(0)}(t, x) = \phi_0(x)e^{it\omega|\phi_0(x)|^{p-1}}.
\]
Then there exist \(C, c > 0\) such that if \(0 < \nu \leq c\) is sufficiently small, then
\[
\|\phi^{(\nu)}(t) - \phi^{(0)}(t)\|_{H^k} \leq C\nu^{2\sigma}
\]
for all \(|t| \leq c|\log \nu|^c\).

Proof. The proof closely follows the proof of Lemma 2.1 in [CCT03a]. □

Obviously, \(\phi^{(\nu)}(t, x)\) is a solution to \((\text{NLS}_\sigma)\). Moreover, \(\phi^{(\nu)}(t, x)\) is bounded and almost flat in the following sense.
Corollary 5.2. Let \( \phi^{(\nu)} \), \( \nu \) and \( c \) be in Lemma 5.1. Let \( s \geq 0 \). Then,
\[
\| \phi^{(\nu)}(t, \nu x) \|_{L^p_\nu} \sim 1
\] (5.4)
and
\[
\| \phi^{(\nu)}(t, \nu x) \|_{\dot{H}^s_\nu} \sim \nu^{s-\frac{d}{2}}(c |\log \nu|^c)^s
\] (5.5)
for all \(|t| \leq c|\log \nu|^c\).

Proof. Since \( k > \frac{d}{2} \), by the Sobolev inequality, we have
\[
\| \phi^{(\nu)}(t, \nu x) - \phi^{(0)}(t, \nu x) \|_{L^p_\nu} = \| \phi^{(\nu)}(t, x) - \phi^{(0)}(t, x) \|_{L^p_\nu} \lesssim \| \phi^{(\nu)}(t) - \phi^{(0)}(t) \|_{H^k} \lesssim \nu^{2\sigma}.
\]
Then, (5.4) follows from the explicit formula (5.2) for \( \phi^{(0)}(t, x) \). It follows from (5.3) and (5.2) that
\[
\| \phi^{(\nu)}(t, \nu x) \|_{\dot{H}^s_\nu} \lesssim \nu^{s-\frac{d}{2}}(\| \phi^{(0)}(t) \|_{\dot{H}^s} + \| \phi^{(\nu)}(t) - \phi^{(0)}(t) \|_{\dot{H}^s}) \sim \nu^{s-\frac{d}{2}}(c |\log \nu|^c)^s.
\]
\[\square\]

For \( v \in \mathbb{R}^d \), we define the pseudo-Galilean transformation by
\[
\mathcal{G}_v u(t, x) = e^{-iv \cdot x} e^{it|v|^{2\sigma}} u(t, x - 2t\sigma|v|^{2(\sigma - 1)} v).
\]
Note that when \( \sigma = 1 \), \( \mathcal{G}_v \) is simply a Galilean transformation, and that NLS is invariant under this transformation, that is, if \( u(t) \) solves NLS, so does \( \mathcal{G}_v u(t) \). However, when \( \sigma \neq 1 \), (NLS\(_\sigma\)) is not exactly symmetric with respect to pseudo-Galilean transformations. Indeed, if \( u(t) \) solves (NLS\(_\sigma\)), then \( \tilde{u}(t) = \mathcal{G}_v u(t) \) obeys (NLS\(_\sigma\)) with an error term
\[
i\hat{\partial}_t \tilde{u} + (-\Delta)^\sigma \tilde{u} + \omega |\tilde{u}|^{p-1} \tilde{u} = e^{it|v|^{2\sigma}} e^{-iv \cdot x} \mathcal{E} u(t, x - 2t\sigma|v|^{2(\sigma - 1)} v),
\] (5.6)
where
\[
\mathcal{E} u(\xi) = E(\xi) \hat{u}(\xi)
\]
with
\[
E(\xi) = |\xi - v|^{2\sigma} - |\xi|^{2\sigma} - |v|^{2\sigma} + 2\sigma|v|^{2(\sigma - 1)} v \cdot \xi.
\]
However, we note that
\[
|E(\xi)| \lesssim |\xi|^{2\sigma}.
\] (5.7)
Indeed, if \( |\xi| \leq \frac{|v|}{100} \), then
\[
E(\xi) = |v|^{2\sigma} \left( |\frac{\xi}{|v|} - \xi|^{2\sigma} - 1 + 2\sigma \frac{\xi}{|v|} \cdot \frac{\xi}{|v|} \right) - |\xi|^{2\sigma} \lesssim |v|^{2\sigma} \frac{|\xi|^2}{|v|^2} + |\xi|^{2\sigma} \lesssim |\xi|^{2\sigma}.
\]
Otherwise,
\[
E(\xi) \lesssim |\xi|^{2\sigma} + |v|^{2\sigma} + |\xi|^{2\sigma} + |v|^{2\sigma} + 2\sigma|v|^{2\sigma - 1}|\xi| \lesssim |\xi|^{2\sigma}.
\]
Therefore, one would expect an *almost* symmetry for an almost flat solution \( u(t) \), such as \( \phi^{(\nu)}(t, \nu x) \) in Lemma 5.1. Precisely, we have the following lemma.

**Lemma 5.3** (Pseudo-Galilean transformation). Let \( \phi^{(\nu)}, \nu \) and \( c \) be in Lemma 5.1. For \( v \in \mathbb{R}^d \), we define

\[
\tilde{u}(t, x) = (\mathcal{G}_c \phi^{(\nu)}(\cdot, \nu \cdot))(t, x) = e^{-i v \cdot x} e^{it|v|^{2\sigma} \phi^{(\nu)}(t, \nu(x - 2t\sigma|v|^{2(\sigma - 1)}v))},
\]

and let \( u(t, x) \) be the solution to \((\text{NLS}_\sigma)\) with the same initial data

\[
e^{-i v \cdot x} \phi^{(\nu)}(0, \nu x) = e^{-i v \cdot x} \phi_0(0, \nu x).
\]

Then, there exists \( \delta > 0 \) such that

\[
\|e^{iv \cdot x}(u(t) - \tilde{u}(t))\|_{H^k_x} \lesssim \nu^\delta
\]

for all \( |t| \leq c|\log \nu|^\epsilon \).

**Remark 5.4.** When \( p = 3 \), in [CHKL14] the authors could use the counterexample in [CCT03b]. This counterexample is constructed by pseudo-conformal symmetry and Galilean transformation. A good thing is that this solution is very small in high Sobolev norms, too. Somehow, this smallness allows [CHKL14] to show that the error in pseudo-Galilean transformation is also small. However, when \( p > 3 \), the counterexample in [CCT03b] does not work. Later, Christ, Colliander and Tao [CCT03a] constructed a different counterexample which works for more general \( p \). Unfortunately, this counterexample is not small in high Sobolev norms. It is very large instead. In particular, for our purposes, it is hard to control the error from pseudo-Galilean transformation. But, our new counterexample still has small high Sobolev norm after translating it to its frequency center; this is the term \( e^{iv \cdot x} \) in equation (5.9). Using this smallness, we can prove that pseudo-Galilean transformation is almost invariant. We also remark that the condition \( \sigma > \frac{d}{4} \) is to guarantee smallness of the error (see (5.10)).

**Proof of Lemma 5.3.** Let \( R(t) = (u - \tilde{u})(t) \). Then, \( R(t) \) satisfies

\[
i \partial_t R + (-\Delta)^\sigma R = \mu(|\tilde{u}|^{p-1} \tilde{u} - |u|^{p-1} u) - e^{it|v|^{2\sigma}} (\mathcal{E} \phi^{(\nu)}(t, \nu(x - 2\sigma|v|^{2(\sigma - 1)}v)),
\]

or equivalently

\[
R(t) = i \int_0^t e^{i(t-s)(-\Delta)^\sigma} \left\{ \mu(|u|^{p-1} u - |\tilde{u}|^{p-1} \tilde{u})(s) + e^{i|v|^{2\sigma}} (\mathcal{E} \phi^{(\nu)}(s, \nu(x - 2\sigma s|v|^{2(\sigma - 1)}v)) \right\} ds.
\]
Hence, by a trivial estimate, we get

\[ \|e^{iv\cdot x}R(t)\|_{H^k} \leq \int_0^t \|e^{iv\cdot x}(|u|^{p-1}u - |\tilde{u}|^{p-1}\tilde{u})(s)\|_{H^k} + \|\mathcal{E}\phi(\nu)(s, \nu, \cdot)\|_{H^k} ds \]

\[ = \int_0^t I(s) + II(s) ds. \]

First, by (5.7) and (5.5), we show that

\[ \int_0^t II(s) ds \lesssim \int_0^t \sum_{j=0}^k \|\phi(\nu)(s, \nu)\|_{H^{j+2\sigma}} ds \sim (c|\log \nu|)^{1+2\sigma - \frac{d}{2}} L^{2\sigma - \frac{d}{2}}. \quad (5.10) \]

For \( I(s) \), expanding \( u = \tilde{u} + R \) and then applying Hölder inequality and Sobolev inequalities, we bound \( I(s) \) by

\[ \lesssim \sum_{j=1}^p \|e^{iv\cdot x}R\|_{H^k}^j. \quad (5.11) \]

For example, when \( p = 3 \),

\[ I(s) \leq 2\|e^{iv\cdot x}\tilde{u}\|^2_{H^k} + \|e^{iv\cdot x}\tilde{u}\|^2_{H^k} + 2\|e^{iv\cdot x}\tilde{u}\|e^{iv\cdot x}R\|_{H^k} \]

\[ + \|e^{iv\cdot x}\tilde{u}\|e^{iv\cdot x}R\|^2_{H^k} + \|e^{iv\cdot x}R\|^2_{H^k} =: I_1(s) + I_2(s) + I_3(s) + I_4(s) + I_5(s). \]

Consider

\[ I_1(s) = \sum_{|\alpha| \leq k} \|\nabla_{x_1}^{\alpha_1} \cdots \nabla_{x_d}^{\alpha_d} (|e^{iv\cdot x}\tilde{u}|^2 e^{iv\cdot x}R)(s)\|_{L^2} =: \sum_{|\alpha| \leq k} I_{1,\alpha}(s), \]

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \) is a multi-index with \( |\alpha| = \sum_{i=1}^d \alpha_i \). Observe that whenever a derivative hits

\[ e^{iv\cdot x}\tilde{u}(s) = e^{j s |\nu|^{2\sigma}} \phi(\nu)(s, \nu(x - 2s|\nu|^{2(\sigma - 1)})) \]

we get a small factor \( \nu \). Hence, after distributing derivatives by the Leibniz rule, the worst term we have in \( I_{1,\alpha}(s) \) is

\[ \|e^{iv\cdot x}\tilde{u}(s)|^2 \nabla^\alpha e^{iv\cdot x}R(s)\|_{L^2} , \]

which is, by (5.4), bounded by

\[ \|e^{iv\cdot x}\tilde{u}(s)|^2 \nabla^\alpha e^{iv\cdot x}R(s)\|_{L^2} \sim \|\nabla^\alpha e^{iv\cdot x}R(s)\|_{L^2}. \]

Likewise, we estimate other terms.

Collecting all,

\[ \|e^{iv\cdot x}R(t)\|_{H^k} \lesssim (c|\log \nu|)^{1+2\sigma - \frac{d}{2}} L^{2\sigma - \frac{d}{2}} + \int_0^t \sum_{j=1}^p \|e^{iv\cdot x}R(s)\|_{H^k}^j ds \]
for $|t| \leq c|\log \nu|^c$. Then, by the standard nonlinear iteration argument, we prove the lemma.

Since we have solutions almost symmetric with respect to the pseudo-Galilean transformations, we can make use of the following decoherence lemma to construct counterexamples for local well-posedness.

**Lemma 5.5 (Decoherence).** Let $s < 0$. Fix a nonzero Schwartz function $w$. For $a, a' \in [\frac{1}{2}, 1]$, $0 < \nu \leq \lambda \ll 1$ and $v \in \mathbb{R}^d$ with $|v| \geq 1$, we define

$$
\tilde{u}^{(a,\nu,\lambda,v)}(t, x) := \mathcal{G}_v \left( \lambda^{-\frac{2\alpha}{p-1}} \phi^{(a,\nu)}(\lambda^{-2\alpha}, \lambda^{-1}\nu) \right)(t, x),
$$

where $\phi^{(a,\nu)}$ is the solution to (5.1) with initial data $aw$. Then, we have

$$
\|\tilde{u}^{(a,\nu,\lambda,v)}(0)\|_{H^s}, \|\tilde{u}^{(a',\nu,\lambda,v)}(0)\|_{H^s} \leq C|v|^s \lambda^{-\frac{2\alpha}{p-1}} \left( \frac{1}{\nu} \right)^{d/2},
$$

$$
\|\tilde{u}^{(a,\nu,\lambda,v)}(0) - \tilde{u}^{(a',\nu,\lambda,v)}(0)\|_{H^s} \leq C|v|^s \lambda^{-\frac{2\alpha}{p-1}} \left( \frac{1}{\nu} \right)^{d/2} |a - a'|
$$

and

$$
\|\tilde{u}^{(a,\nu,\lambda,v)}(t) - \tilde{u}^{(a',\nu,\lambda,v)}(t)\|_{H^s}
\geq c|v|^s \lambda^{-\frac{2\alpha}{p-1}} \left( \frac{1}{\nu} \right)^{d/2} \left\{ \|\phi^{(a,\nu)}(\frac{t}{\lambda^{2\alpha}}) - \phi^{(a',\nu)}(\frac{t}{\lambda^{2\alpha}})\|_{L^2} - C|\log \nu|^C \left( \frac{1}{\nu} \right)^{-k} |v|^{-s-k} \right\}
$$

for all $|t| \leq c|\log \nu|^c \lambda^{2\alpha}$.

**Proof.** The proof closely follows the proof of Lemma 3.1 in [CCT03a].

**Proof of Theorem 1.5.** The proof is very similar to that of Theorem 1 in [CCT03a] except that in the last step, we need to use Lemma 5.3 due to lack of exact symmetry. We give a proof for the readers' convenience.

Let $\epsilon > 0$ be a given but arbitrarily small number. Let $\nu = \lambda^a$, where $0 < \alpha > 0$ is a small number to be chosen later. Then, we pick $v \in \mathbb{R}^d$ such that

$$
\lambda^{-\frac{2\alpha}{p-1}} |v|^s \left( \frac{1}{\lambda^d} \right)^{d/2} = \epsilon \Leftrightarrow |v| = \nu^{\frac{1}{s} \left( \frac{d(1-\alpha)}{2} + \frac{2\alpha}{p-1} \right)} \epsilon^{1/s}.
$$

Note that since $s < 0$, $\frac{1}{s} \left( \frac{d(1-\alpha)}{2} + \frac{2\alpha}{p-1} \right) = \frac{1}{s} \left( \frac{d}{2} - \alpha s \right) < 0$ for sufficiently small $\alpha$, and thus $|v| \geq 1$. Hence, it follows from Lemma 5.5 that

$$
\|\tilde{u}^{(a,\nu,\lambda,v)}(0)\|_{H^s}, \|\tilde{u}^{(a',\nu,\lambda,v)}(0)\|_{H^s} \leq C \epsilon,
$$

$$
\|\tilde{u}^{(a,\nu,\lambda,v)}(0) - \tilde{u}^{(a',\nu,\lambda,v)}(0)\|_{H^s} \leq C \epsilon |a - a'|,
$$

and

$$
\|\tilde{u}^{(a,\nu,\lambda,v)}(t) - \tilde{u}^{(a',\nu,\lambda,v)}(t)\|_{H^s}
\geq c \epsilon \left\{ \|\phi^{(a,\nu)}(\frac{t}{\lambda^{2\alpha}}) - \phi^{(a',\nu)}(\frac{t}{\lambda^{2\alpha}})\|_{L^2} - C|\log \nu|^C \left( \frac{1}{\nu} \right)^{-k} |v|^{-s-k} \right\}
$$
for all $|t| \leq c \log \nu |\lambda|^{2\sigma}$. Now we observe from the explicit formula (5.2) for $\phi^{(a,0)}$ and (5.1) that there exists $T > 0$ such that $\|\phi^{(a,\nu)}(T) - \phi^{(a',\nu)}(T)\|_{L^2} \geq c$. Moreover, if $\alpha > 0$ is sufficiently small, $C \log \nu |\frac{1}{\nu}|^k |v|^{-s-k} \to 0$ as $\nu \to 0$. Therefore, for $\nu$ small enough, we have

$$\|\tilde{u}^{(a,\nu,\lambda,v)}(\lambda^{2\sigma} T) - \tilde{u}^{(a',\nu,\lambda,v)}(\lambda^{2\sigma} T)\|_{H^s} \geq c\epsilon.$$  (5.14)

Next, we replace $\tilde{u}^{(a,\nu,\lambda,v)}$ and $\tilde{u}^{(a',\nu,\lambda,v)}$ in (6.11), (6.12) and (6.13) by $u^{(a,\nu,\lambda,v)}$ and $u^{(a',\nu,\lambda,v)}$ by Lemma 5.3 with $O(\nu^3)$ error. Then, making $|a - a'|$ arbitrarily small and then sending $\nu \to 0$ (so, $\lambda^{2\sigma} T \to 0$), we complete the proof.

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