On the Black's equation for the risk tolerance function*

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Abstract

We analyze a nonlinear equation proposed by F. Black (1968) for the optimal portfolio function in a log-normal model. We cast it in terms of the risk tolerance function and provide, for general utility functions, existence, uniqueness and regularity results, and we also examine various monotonicity, concavity/convexity and S-shape properties. Stronger results are derived for utilities whose inverse marginal belongs to a class of completely monotonic functions.

1 Introduction

In 1968[1], Fisher Black discovered an autonomous equation, which he called the "*investment equation*", that the optimal portfolio function satisfies, as a function of wealth, time and the optimal consumption in a log-normal model (see [6]). With the notation used therein, the equation is

$$x_2 = (r - c_1)\, x - (wr - c)\, x_1 - \frac{1}{2} s^2 x^2 x_{11}, \tag{1}$$

where x is the optimal investment function (the subscripts $1, 2$ correspond to partial derivatives in space and time) and w the wealth argument, c is the optimal consumption function, and r, s given market parameters.

Black's equation was much later independently (re)discovered in [23] for a problem examining the compatibility of the optimal consumption function and the utility function, and was also used in [25] where turnpike (long-term) properties of the optimal portfolio functions were examined without intermediate

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consumption. More recently, it has been used in [1], [4], [17], [49], [53], and others.

The aim herein is to provide a systematic study of the above equation in a multi-stock log-normal model without intermediate consumption. We cast it in terms of the local risk tolerance function $r(x,t)$ which, in a trading horizon $[0,T]$, takes the form

$$r_t + \frac{1}{2}|\lambda|^2 r^2 r_{xx} = 0,$$

$(x,t) \in \mathbb{R}_+ \times [0,T)]$, with $r(x,T) = R(x)$, the risk tolerance coefficient, and λ being the market price of risk (cf. Proposition 3). Throughout, we will refer to this nonlinear equation as the *Black's equation* for the risk tolerance function.

A related nonlinear equation is satisfied by the reciprocal $\gamma(x,t) = \frac{1}{r(x,t)}$, known as the local risk aversion function. It then follows that $\gamma(x,t)$ solves a *porous medium* equation (PME),

$$\gamma_t - \frac{1}{2}\left(\gamma^{-1}\right)_{xx} = 0.$$

The PME is typically classified by its "exponent" m in the standard representation $\gamma_t - \frac{1}{2}(\gamma^m)_{xx} = 0$. In our case, $m = -1$, which corresponds to the so-called *fast diffusion* regime ($m < 1$); see, for example, (see [54]).

Our contribution is multi-fold. We study the existence, uniqueness and regularity of the solution to the Black's equation and provide estimates for its derivatives. We also study its spatial monotonicity, concavity/convexity and S-shaped properties. Specifically, we investigate when such properties satisfied by the risk tolerance coefficient $R(x)$ are inherited to $r(x,t)$, for all times $t \in [0,T)$. We also study the time-monotonicity of $r(x,t)$ as well as its dependence on $|\lambda|^2$. Furthermore, we provide analogous results for the local relative risk tolerance function, $\tilde{r}(x,t) = \frac{r(x,t)}{x}$.

With the exception of the uniqueness result, for all other ones pivotal role plays a function H that solves the heat equation, appearing through the nonlinear transformation,

$$r(H(z,t),t) = H_x(z,t), \tag{2}$$

$(z,t) \in \mathbb{R}\times [0,T]$.

This transformation results from a variation of the Legendre-Fenchel one (applied to the value function), frequently used in stochastic optimization with linear control dynamics. However, the form we propose is much more convenient, for it simplifies a number of highly nonlinear expressions we want to analyze. Indeed, these expressions reduce to much simpler ones involving the partial derivatives of the aforementioned harmonic function. Like H, these derivatives also solve the heat equation and, thus, classical results can be in turn applied like the maximum principle, the preservation of the log-concavity/convexity of its solutions, the properties of their zero points sets, and others.

We note that this transformation was first proposed in an Ito-diffusion setting in [41] for a different class of risk preferences, the so-called time-monotone forward performance processes. Therein, the corresponding harmonic function satisfies the ill-posed heat equation, and has quite different characteristics. Nevertheless, various expressions and equations are algebraically similar but the nature of the solutions is fundamentally different.

To show uniqueness, we do not use (2) but work directly with a semilinear equation (cf. (31)) satisfied by $r^2(x,t)$. In turn, the uniqueness in combination with the convexity/concavity results yields the monotonicity of the risk tolerance function with respect to $|\lambda|^2$.

Some of the results we provide, namely, the monotonicity and the concavity, have been shown before (see [8]) but we provide considerably shorter and more direct alternative proofs, bypassing various lengthy arguments. The uniqueness was also established in [53] using duality techniques, while we provide a much shorter proof based on PDE arguments. The regularity results as well as the ones on the time monotonicity and the S-properties are, to the best of our knowledge, new.

Besides the results for general risk preferences, we examine the class of utilities whose inverse marginal I is a completely monotonic function of a given form, namely, $I(x) = \int_\alpha^\beta x^{-y} d\mu(y)$, $0 < \alpha < \beta < \infty$, and μ a finite positive Borel measure. For such cases, stronger bounds and regularity estimates can be derived, which we provide.

We work in a multi-stock log-normal model. While this is a simple market setting, we nevertheless gain a number of valuable insights for the risk tolerance function and its derivatives that were not known before, and also provide much shorter and direct proofs for existing results.

Beyond the log-normal model, our results can be also used in more general diffusion settings for the analysis of the zeroth order term in stochastic factor models with slow and fast factors (see, for example, [1], [16], [17], [36], [37]). In these models, this term is similar to the value function (9) herein but with rescaled deterministic time.

The paper is organized as follows. In section 2 we introduce the investment model and provide background and auxiliary results. In section 3 we derive the risk tolerance equation, and study the uniqueness and regularity of its solutions. In section 4, we provide further properties and conclude in section 5 with the example of a family of completely monotonic inverse marginals.

2 The model and preliminary results

We start, for the reader's convenience, with a brief review of the classical Merton problem ([39]). Investment takes place on $[0,T]$, a given trading horizon. The market environment consists of one riskless and N risky securities. The risky securities are stocks and their prices are modelled as log-normal processes. Namely, for $i = 1, ..., N$, the price S_t^i, $0 \le t \le T$, of the i^{th} risky asset satisfies

$$dS_t^i = S_t^i \left(\mu^i dt + \sum_{j=1}^{N} \sigma^{ji} dW_t^j \right), \tag{3}$$

with $S_0^i > 0$. The process $W_t = \left(W_t^1, ..., W_t^N\right), t \geq 0$, is a standard $N-$dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $\mathcal{F}_t = \sigma\left(W_s: \ 0 \leq s \leq t\right)$.

The constants μ^i and $\sigma^i = \left(\sigma_t^{1i}, ..., \sigma^{Ni}\right), i = 1, ..., N, \ t \geq 0$, take values in $\mathbb{R}$ and $\mathbb{R}^N$, respectively. For brevity, we use σ to denote the $N \times N$ matrix volatility $\left(\sigma^{ji}\right)$, whose i^{th} column represents the volatility $\sigma^i = \left(\sigma^{1i}, ..., \sigma^{Ni}\right)$ of the i^{th} risky asset. Alternatively, we write equation (3) as

$$dS_t^i = S_t^i \left(\mu^i dt + \sigma^i \cdot dW_t\right).$$

The riskless asset, the savings account, offers constant interest rate $r > 0$. We denote by μ the $N \times 1$ vector with coordinates μ^i and by $\mathbf{1}$ the $N-$dimensional vector with every component equal to one. We assume that the volatility matrix is invertible, and define the vector

$$\lambda = \left(\sigma^T\right)^{-1} \left(\mu - r\mathbf{1}\right). \tag{4}$$

Starting at $t \in [0, T)$ with initial endowment $x > 0$, the investor invests at any time $s \in (t, T]$ in the riskless and risky assets. The present value of the amounts invested are denoted, respectively, by π_s^0 and π_s^i, $i = 1, ..., N$, and are taken to be self-financing. The (present) value of her investment is, then, given by $X_s^\pi = \sum_{k=0}^N \pi_s^k$, $s \in (t, T]$, which solves

$$dX_s^\pi = \sigma\pi_s \cdot \left(\lambda ds + dW_s\right), \tag{5}$$

with $X_t = x$, and where the (column) vector, $\pi_s = \left(\pi_s^i; \ i = 1, ..., N\right)$.

A self-financing investment process π_s is admissible if $\pi_s \in \mathcal{F}_s$, $E_{\mathbb{P}}\left(\int_t^T \left|\pi_s\right|^2 ds\right) < \infty$ and the associated wealth remains non-negative, $X_s^\pi \geq 0, \ \ s \in [t, T]$. We denote the set of admissible strategies by $\mathcal{A}$.

The utility function at T is given by $U : \mathbb{R}_+ \to \mathbb{R}$, and it is assumed to be a strictly concave, strictly increasing and $C^\infty(0, \infty)$ function, satisfying the Inada conditions $\lim_{x \downarrow 0} U'(x) = \infty$ and $\lim_{x \uparrow \infty} U'(x) = 0$.

We recall the inverse marginal $I : \mathbb{R}_+ \to \mathbb{R}_+$, $I(x) = (U')^{(-1)}(x)$. It is assumed that it satisfies, for $C, \delta > 0$,

$$I(x) \leq C\left(1 + x^{-\delta}\right), \tag{6}$$

and for positive constants c_n, C_n, $n = 1, 2, 3$, with $c_2 > 1$,

$$c_1 I(x) \leq \left|xI'(x)\right| \leq C_1 I(x), \tag{7}$$

$$c_2\left|I'(x)\right| \leq xI''(x) \leq C_2\left|I'(x)\right| \quad \text{and} \quad \left|xI'''(x)\right| \leq C_3 I''(x). \tag{8}$$

The above conditions are rather mild and satisfied by a large class of utility functions. For example, if $I(x) = \Sigma_{i=1}^N x^{-\alpha_i}$, $0 < \alpha_1 < ..., \alpha_N$, (7) and (8) hold for $c_1 = \alpha_1, C_1 = \alpha_N, c_2 = \alpha_1 + 1, C_2 = \alpha_N + 1, C_3 = \alpha_N + 2$.

Throughout, we will use the domain notation $\mathbb{D}_+ = \mathbb{R}_+ \times [0, T]$ and $\mathbb{D} = \mathbb{R} \times [0, T]$.

The value function $u : \mathbb{D}_+ \rightarrow \mathbb{R}_+$ is defined as the maximal expected utility of terminal wealth,

$$u(x,t) = \sup_{\pi \in \mathcal{A}} E_{\mathbb{P}} \left(U \left(X_T^{\pi} \right) \middle| \, X_t^{\pi} = x \right), \tag{9}$$

where X_s^{π}, $s \in (t, T]$, solves (5). This optimization problem has been extensively studied (see, for example, [5] and [29]). It is known that $u \in C^{\infty,1}(\mathbb{D}_+)$, it is strictly increasing and strictly concave in the spatial variable, and solves the Hamilton-Jacobi-Bellman (HJB) equation,

$$u_t - \frac{1}{2} |\lambda|^2 \frac{u_x^2}{u_{xx}} = 0, \tag{10}$$

with $u(x,T) = U(x)$ and λ as in (4).

The absolute risk tolerance coefficient $R(x)$ and the local risk tolerance function $r(x,t)$ are defined, respectively, by

$$R(x) = -\frac{U'(x)}{U''(x)} \qquad \text{and} \quad r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{11}$$

for $(x,t) \in \mathbb{D}_+$. A standing assumption is that $R(0) = 0$ and that $R(x)$ is strictly increasing (see [3]). The latter implies that $I''(x) > 0$, $x > 0$.

To ease the presentation, we eliminate the terminology "absolute" and "local".

Next we introduce the function H which plays a crucial role herein and provide auxiliary results for it and its spatial derivatives.

Lemma 1 *Let $H : \mathbb{D} \rightarrow \mathbb{R}_+$ be defined by*

$$u_x(H(z,t), t) = e^{-z - \frac{1}{2}|\lambda|^2 (T-t)}, \tag{12}$$

where $u(x,t)$ is as in (9). Then, $H(z,t) \in C^{\infty,1}(\mathbb{D})$ and solves the heat equation

$$H_t + \frac{1}{2} |\lambda|^2 H_{zz} = 0, \tag{13}$$

with terminal condition

$$H(z,T) = I\left(e^{-z}\right). \tag{14}$$

For each $t \in [0,T]$, it is strictly increasing and of full range,

$$\lim_{z \downarrow -\infty} H(z,t) = 0 \qquad \text{and} \qquad \lim_{z \uparrow \infty} H(z,t) = \infty. \tag{15}$$

Furthermore, the risk tolerance coefficient satisfies

$$R(H(z,T)) = H_z(z,T). \tag{16}$$

Proof. The fact that H is well defined follows from the spatial invertibility of u. The regularity of H and the fact that it solves the heat equation follows directly from (10) and (12).

The existence and uniqueness of solutions to (13) follow from the terminal datum (14) and property (6) which yields $H(x,T) \leq C\left(1+e^{\delta x}\right)$ (see, for example, [44], [52]).

The monotonicity of H follows from the strict spatial concavity of u. To show (15), we use (12) and that the value function $u(x,t)$ satisfies, for $t \in [0,T]$, the Inada conditions (see, for example, [29]). Equality (16) follows directly from (12) and (11). ■

Lemma 2 *Assume inequalities (6), (7) and (8) hold, and that H solves (13) and (14). Then, for $(z,t) \in \mathbb{D}$ the following assertions hold for some positive constants M_n, n_n, N_n, $n = 1,2,3$.*

i) The functions $\frac{\partial^n H(z,t)}{\partial z^n}$ solve the heat equation (13) with

$$0 < \frac{\partial^n H(z,T)}{\partial z^n} \leq M_n(1+e^{\delta z}),\ n = 1,2, \tag{17}$$

and

$$\left|\frac{\partial^3 H(z,T)}{\partial z^3}\right| \leq M_3(1+e^{\delta z}), \tag{18}$$

where δ as in (6).

ii) The following inequalities hold

$$n_n \frac{\partial^{n-1} H(z,t)}{\partial z^{n-1}} \leq \frac{\partial^n H(z,t)}{\partial z^n} \leq N_n \frac{\partial^{n-1} H(z,t)}{\partial z^{n-1}},\ n = 1,2, \tag{19}$$

and

$$\left|H_{zzz}(z,t)\right| \leq N_3 \left|H_{zz}(z,t)\right|. \tag{20}$$

Proof. i) The fact that the partial derivatives $\frac{\partial^n H(z,t)}{\partial z^n}$ solve the heat equation follows directly from (13). To show (17) recall that $H_z(z,T) = -e^{-z} I'(e^{-z}) > 0$. Furthermore, (14) and (6) yield

$$H_z(z,T) = \left|e^{-z} I'\left(e^{-z}\right)\right| \leq C_1 I\left(e^{-z}\right) \leq C_1 C(1+e^{\delta z}),$$

where we used (6) and (7).

For $n = 2$, observe that (16) yields $H_z(z,T) R'(H(z,T)) = H_{zz}(z,T)$. Using the assumption that $R' > 0$ and the full range of H, we deduce that $H_{zz}(z,T) > 0$. Furthermore,

$$H_{zz}(z,T) = e^{-z} I'\left(e^{-z}\right) + e^{-2z} I''\left(e^{-z}\right) < e^{-2z} I''\left(e^{-z}\right)$$

$$\leq C_2 \left|e^{-z} I'\left(e^{-z}\right)\right| \leq C_2 C_1 C(1+e^{\delta z}),$$

where we used (7), (8) and (6). Inequality (18) follows similarly.

ii) Because of inequalities (17) and (18), we have that comparison holds for the heat equation satisfied by H_z, H_{zz} and H_{zzz}. Therefore, to show (19) and

(20), it suffices to establish them for $t = T$ only, which follows from direct differentiation of (14) and repeated use of (7) and (8). ■

We note that similar bounds to (18) for higher order partial derivatives, $\frac{\partial^n H(z,T)}{\partial z^n}$, $n > 3$, can be obtained if one imposes analogous to (7) and (8) inequalities for the partial derivatives $I^{(n)}(x)$. Then, using that $\frac{\partial^n H(z,t)}{\partial z^n}$ also satisfy the heat equation, we can deduce analogous to (20) bounds. These results can be strengthened if further information about the sign of $\frac{\partial^n H(z,T)}{\partial z^n}$ is known. We revert to such cases in section 5.

3 Black's equation for the risk tolerance

We start with the derivation of the Black's equation, and study questions on existence, uniqueness and regularity of its solution.

Proposition 3 *Let $r(x,t)$ be the risk tolerance function and $H(x,t)$ the solution to the heat question (13) and (14). Then, the following assertions hold.*

i) For $(x,t) \in \mathbb{D}_+$,

$$r(x,t) = H_z\left(H^{(-1)}(x,t), t\right). \tag{21}$$

ii) Furthermore, $r(x,t) \in C^{\infty,1}(\mathbb{D}_+)$ and solves the nonlinear equation

$$r_t + \frac{1}{2}|\lambda|^2 r^2 r_{xx} = 0, \tag{22}$$

with $r(x,T) = R(x)$ and $r(0,t) = 0$, $t \in [0,T]$.

Proof. Property (21) follows directly from (11) and transformation (12). To show (22), we have from (21) that

$$r(H(z,t), t) = H_z(z,t). \tag{23}$$

Differentiating twice yields, with all arguments of H and its derivatives evaluated at (z,t),

$$H_t r_x(H,t) + r_t(H,t) = H_{zt}, \quad H_z r_x(H,t) = H_{zz}$$

and

$$H_{zz} r_x(H,t) + H_z^2 r_{xx}(H,t) = H_{zzz}.$$

Using that H_z solves (13) we deduce

$$H_t r_x(H,t) + r_t(H,t) + \frac{1}{2}\left(H_{zz} r_x(H,t) + H_z^2 r_{xx}(H,t)\right) = 0,$$

and rearranging terms we obtain

$$r_t(H,t) + \frac{1}{2}|\lambda|^2 H_z^2 r_{xx}(H,t) + r_x(H,t)\left(H_t + \frac{1}{2}|\lambda|^2 H_{zz}\right) = 0.$$

From (13) we then get

$$r_t(H,t) + \frac{1}{2}|\lambda|^2 r^2(H,t)\, r_{xx}(H,t) = 0,$$

and using that, for each t, the function $H(\cdot,t)$ is of full range we conclude. For the values of $r(0,t)$, see [53]. ■

Next, we derive various regularity estimates on the derivatives of the risk tolerance and of the ratio $r(x,t)/x$. Other estimates may be found in [16].

Proposition 4 *Assume that the inverse marginal utility function I satisfies (7) and (8). Then, there exist positive constants $k_n, l_n, K_n, L_n, K, m, M$, such that the following assertions hold.*

i) For $n = 0, 1$,

$$k_n \le x^{n-1}\frac{\partial^n r(x,t)}{\partial x^n} \le K_n \quad \text{and} \quad |x r_{xx}(x,t)| \le K_2, \tag{24}$$

and, for $n = 0, 1, 2$,

$$l_n \le \left| x^n \frac{\partial^n}{\partial x^n}\left(\frac{r(x,t)}{x}\right)\right| \le L_n. \tag{25}$$

Furthermore,

$$\left|r_{xx}^2(x,t)\right| \le K. \tag{26}$$

ii) For each $x > 0$, $t \in [0,T)$,

$$mx \le |r_t(x,t)| \le Mx. \tag{27}$$

Proof. i) We first show (24) for $n = 0$. From (23) and (19) we have

$$\frac{r(H(z,t),t)}{H(z,t)} = \frac{H_z(z,t)}{H(z,t)} \le N_1,$$

and the upper bound follows setting $K_0 = N_1$. The lower bound follows similarly.

Moreover, from (21) and (19) we deduce

$$r_x(H(z,t),t) = \frac{H_{zz}(z,t)}{H_z(z,t)} \le N_2,$$

and thus the upper bound in (24) holds for $K_1 = N_2$.

For $n = 2$,

$$H_z(z,t)\, r_{xx}(H(z,t),t) = \frac{H_{zzz}(z,t)}{H_z(z,t)} - \left(\frac{H_{zz}(z,t)}{H_z(z,t)}\right)^2 \tag{28}$$

and thus

$$|H(z,t)\, r_{xx}(H(z,t),t)| \le \frac{H(z,t)}{H_z(z,t)}\left(\left|\frac{H_{zzz}(z,t)}{H_z(z,t)}\right| + \left(\frac{H_{zz}(z,t)}{H_z(z,t)}\right)^2\right)$$

and the bound follows easily.

For (26), it suffices to observe, using (23), that

$$r_{xx}^{2}\left(H\left(z,t\right),t\right)=2\frac{H_{zzz}\left(z,t\right)}{H_{z}\left(z,t\right)},$$

and we easily conclude using (19) and (20).

ii) From equation (22) and equality (28) above, we have

$$\begin{aligned} r_{t}\left(H,t\right) &= -\frac{1}{2}r\left(H,t\right)H_{z}r_{xx}\left(H,t\right) \\ &= -\frac{1}{2}H_{z}\left(\frac{H_{zzz}}{H_{z}}-\left(\frac{H_{zz}}{H_{z}}\right)^{2}\right). \end{aligned}$$

Thus,

$$\left|\frac{r_{t}\left(H,t\right)}{H}\right|\leq\frac{1}{2}\frac{H_{z}}{H}\left(\left|\frac{H_{zzz}}{H_{z}}\right|+\left(\frac{H_{zz}}{H_{z}}\right)^{2}\right),$$

and the upper bound in (27) follows. ■

3.1 Uniqueness of solutions

Formula (21) yields the existence of smooth solutions to the risk tolerance equation (22). We next investigate the uniqueness.

To our knowledge, this question has been investigated only by Xia in [53], using an approximating sequence of penalized versions of equation (22) and duality arguments to obtain comparison of their solutions. As he mentions (see Remark 4.3 in [53]), it is quite difficult to obtain comparison results directly from the equation itself.

Herein, we provide such a result. The key idea is to consider an auxiliary equation, specifically, the one satisfied by the square of the risk tolerance function (cf. (31)), and establish comparison for this equation instead. The comparison result for (22) would then follow using the positivity of the risk tolerance functions.

Proposition 5 *Let I_1, I_2 be inverse marginal utility functions satisfying (6),(7) and (8), and let R_1 and R_2 be the associated risk tolerance coefficients, satisfying, for $x \geq 0$,*

$$R_{1}\left(x\right)\leq R_{2}\left(x\right). \tag{29}$$

Then, for $(x,t) \in \mathbb{D}_{+}$,

$$r_{1}\left(x,t\right)\leq r_{2}\left(x,t\right), \tag{30}$$

with r_1, r_2 solving (22), with $r_1(x,T) = R_1(x), r_2(x,T) = R_2(x)$.

Proof. We first observe that equation (22) yields that $F := r^2$ solves the semilinear equation

$$F_t + \frac{1}{2} |\lambda|^2 F F_{xx} - \frac{1}{4} |\lambda|^2 F_x^2 = 0, \tag{31}$$

$(x,t) \in \mathbb{D}_+$, with $F(x,T) = R^2(x)$. To facilitate the exposition we will work with $\bar{F}(x,t) := F(x, T-t)$ instead. Then,

$$\bar{F}_t - \frac{1}{2} |\lambda|^2 \bar{F} \bar{F}_{xx} + \frac{1}{4} |\lambda|^2 \bar{F}_x^2 = 0, \tag{32}$$

with $\bar{F}(x,0) = R^2(x)$. Therefore, the functions $f(x,t)$ and $g(x,t)$ defined as

$$f(x,t) := r_1^2(x, T-t) \quad \text{and} \quad g(x,t) := r_2^2(x, T-t)$$

are (sub-) and (super-) solutions of (32),

$$f_t - \frac{1}{2} |\lambda|^2 f f_{xx} + \frac{1}{4} |\lambda|^2 f_x^2 \leq 0 \;\text{ and }\; g_t - \frac{1}{2} |\lambda|^2 g g_{xx} + \frac{1}{4} |\lambda|^2 g_x^2 \geq 0,$$

with

$$f(0,t) = g(0,t) = 0 \quad \text{and} \quad f(x,0) \leq g(x,0).$$

Furthermore, (26) yields, for $(x,t) \in \mathbb{D}_+$,

$$f_{xx}(x,t) \leq K \quad \text{and} \quad g_{xx}(x,t) \leq K. \tag{33}$$

We are going to establish that, for $t \in (0,T)$, $x > 0$,

$$f(x,t) \leq g(x,t). \tag{34}$$

For this, we follow parts of the proof of Theorem 3.1 in [18]. To this end, let

$$m = \begin{cases} T, & \text{if } T < \frac{1}{|\lambda|^2 \left(6K_1^2 + \frac{1}{2}K\right)} \\ m_0 \in \left(0, \frac{1}{|\lambda|^2 \left(6K_1^2 + \frac{1}{2}K\right)}\right), & \text{otherwise} \end{cases}, \tag{35}$$

with K_1, K as in (24) and (26), respectively. Then, $m \leq T$.

Consider the domain $\mathbb{D}_+^m = \mathbb{R}_+ \times [0,m]$ and introduce, for $(x,t) \in \mathbb{D}_+^m$, the auxiliary functions

$$f_m(x,t) := \left(1 - \frac{t}{m}\right) f(x,t) \quad \text{and} \quad g_m(x,t) := \left(1 - \frac{t}{m}\right) g(x,t). \tag{36}$$

As argued in [18], a bootstrapping argumentation can be used to establish (34) once it is shown that, for $(x,t) \in \mathbb{D}_+^m$, the inequality

$$f_m(x,t) \leq g_m(x,t) \tag{37}$$

holds. Next, observe that, for $(x,t) \in \mathbb{D}_+^m$, we have

$$0 \le f_m(x,t) \le f(x,t) \qquad \text{and} \qquad 0 \le g_m(x,t) \le g(x,t). \tag{38}$$

Moreover,

$$f_m(x,0) \le g_m(x,0) \quad \text{and} \quad f_m(x,m) = g_m(x,m) = 0, \tag{39}$$

and

$$f_m(0,t) = g_m(0,t) = 0. \tag{40}$$

By the assumption that $r_1(x,t)$ and $r_2(x,t)$ are (sub- and super-) solutions of (22), and the definition of f_m and g_m, we get

$$\left(1 - \frac{t}{m}\right) f_{m,t} + \frac{1}{m} f_m - \frac{1}{2} |\lambda|^2 f_m f_{m,xx} + \frac{1}{4} |\lambda|^2 f_{m,x}^2 \le 0 \tag{41}$$

and

$$\left(1 - \frac{t}{m}\right) g_{m,t} + \frac{1}{m} g_m - \frac{1}{2} |\lambda|^2 g_m g_{m,xx} + \frac{1}{4} |\lambda|^2 g_{m,x}^2 \ge 0. \tag{42}$$

We now establish (37). To this end, we consider the test function $\varphi(x) = 1 + x^4$, $x \ge 0$, and show that, for any $\varepsilon > 0$,

$$f_m(x,t) \le g_m(x,t) + \varepsilon \varphi(x).$$

We argue by contradiction, assuming that there exists $\varepsilon > 0$ such that

$$\sup_{(x,t) \in \mathbb{D}_+} \left(f_m(x,t) - g_m(x,t) - \varepsilon \varphi(x)\right) > 0. \tag{43}$$

Let (x,t) be any point such that $f_m(x,t) - g_m(x,t) - \varepsilon\varphi(x) > 0$. Using that $r_i(x,t) \le K_1 x$ (cf. (24)) and inequalities (38), we deduce that $h(x,t) \le K_1^2 x^2$, for $h = f_m, g_m$, which together with the growth of φ yields that $x < \infty$. Furthermore, we observe that the extremum in (43), denoted by $(\tilde{x}, \tilde{t})$, is an interior point in $\mathbb{D}_+^m$. Indeed, if $(\tilde{x}, \tilde{t})$ is such that $\tilde{t} = 0$ or $\tilde{t} = m$ we get a contradiction from (39), while if, for some $\tilde{t} \in (0,m)$, $(\tilde{x}, \tilde{t}) = (0, \tilde{t})$, we contradict (40).

At the interior maximum $(\tilde{x}, \tilde{t})$ in (43) we then have

$$f_m(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) > \varepsilon\left(1 + \tilde{x}^4\right) \tag{44}$$

$$f_{m,t}(\tilde{x}, \tilde{t}) - g_{m,t}(\tilde{x}, \tilde{t}) = 0, \quad f_{m,x}(\tilde{x}, \tilde{t}) - g_{m,x}(\tilde{x}, \tilde{t}) = 4\varepsilon\tilde{x}^3 \tag{45}$$

and

$$f_{m,xx}(\tilde{x}, \tilde{t}) - g_{m,xx}(\tilde{x}, \tilde{t}) \le 12\varepsilon\tilde{x}^2. \tag{46}$$

From (41) and (42) we deduce

$$\frac{1}{m}\left(f_m(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t})\right) \le \frac{1}{2}|\lambda|^2 \left(f_m(\tilde{x}, \tilde{t}) f_{m,xx}(\tilde{x}, \tilde{t}) - g_m(\tilde{x}, \tilde{t}) g_{m,xx}(\tilde{x}, \tilde{t})\right)$$

$$-\frac{1}{4}|\lambda|^2\left(f_{m,x}^2\left(\tilde{x},\tilde{t}\right)-g_{m,x}^2\left(\tilde{x},\tilde{t}\right)\right).$$

In turn,

$$\left(\frac{1}{m}-\frac{1}{2}|\lambda|^2 f_{m,xx}\left(\tilde{x},\tilde{t}\right)\right)\left(f_m\left(\tilde{x},\tilde{t}\right)-g_m\left(\tilde{x},\tilde{t}\right)\right)$$

$$\leq\frac{1}{2}|\lambda|^2 g_m\left(\tilde{x},\tilde{t}\right)\left(f_{m,xx}\left(\tilde{x},\tilde{t}\right)-g_{m,xx}\left(\tilde{x},\tilde{t}\right)\right)$$

$$-\frac{1}{4}|\lambda|^2\left(f_{m,x}\left(\tilde{x},\tilde{t}\right)-g_{m,x}\left(\tilde{x},\tilde{t}\right)\right)\left(f_{m,x}\left(\tilde{x},\tilde{t}\right)+g_{m,x}\left(\tilde{x},\tilde{t}\right)\right).$$

Inequality (26) and the definition of f_m yield that $f_{m,xx}(x,t)<K$. This, in combination with $f_m\left(\tilde{x},\tilde{t}\right)-g_m\left(\tilde{x},\tilde{t}\right)>0$ (cf. (44)) and (45) and (46), implies that

$$\left(\frac{1}{m}-\frac{1}{2}|\lambda|^2 K\right)\left(f_m\left(\tilde{x},\tilde{t}\right)-g_m\left(\tilde{x},\tilde{t}\right)\right)$$

$$\leq 6\varepsilon|\lambda|^2 g_m\left(\tilde{x},\tilde{t}\right)\tilde{x}^2-\varepsilon|\lambda|^2\tilde{x}^3\left(f_{m,x}\left(\tilde{x},\tilde{t}\right)+g_{m,x}\left(\tilde{x},\tilde{t}\right)\right).$$

Using that the functions $r_1(x,t)$ and $r_2(x,t)$, and in turn $f_m(x,t)$ and $g_m(x,t)$, are strictly increasing, the above inequality yields that, at $\left(\tilde{x},\tilde{t}\right)$, we must have

$$\varepsilon\left(\frac{1}{m}-\frac{1}{2}|\lambda|^2 K\right)\left(1+\tilde{x}^4\right)\leq 6\varepsilon|\lambda|^2 g_m\left(\tilde{x},\tilde{t}\right)\tilde{x}^2.$$

Finally, using once more that $g_m\left(\tilde{x},\tilde{t}\right)\leq K_1^2\tilde{x}^2$, the above inequality then yields that $\frac{1}{m}-\frac{1}{2}|\lambda|^2 K\leq 6|\lambda|^2 K_1^2$, and thus we must have

$$m\geq\frac{1}{|\lambda|^2\left(6K_1^2+\frac{1}{2}K\right)},$$

which, however, contradicts the choice of m in (35). ∎

We note that the property $r_{xx}^2(x,t)\leq K$ $(K>0)$, played a crucial role in the above proof. Such functions are frequently called semi super-harmonic.

We also note that in [53] the admissible class of risk tolerance functions satisfy $R(x)\leq M(1+x)$, for $x\geq 0$ and $M>0$. This property allows for risk tolerances with $\lim_{x\downarrow 0}R'(x)=\infty$, which however are excluded herein. This property makes our admissible class slightly smaller than the one in [53]. Such a case is, for example, $R(x)=\sqrt{x}$.

A direct consequence of the above comparison result is the monotonicity of the risk tolerance function on the market parameter $|\lambda|^2$. Note that while the terminal condition $r(x,T)=R(x)$ is independent on $|\lambda|^2$, this is not the case for $t<T$, since $r(x,t)=-\frac{u_x(x,t)}{u_{xx}(x,t)}$, with both $u_x(x,t)$ and $u_{xx}(x,t)$ depending on $|\lambda|^2$ (cf. (10)).

As we demonstrate below, the dependence of $r(x,t)$ on $|\lambda|^2$ for *all* times depends exclusively on the *curvature* of the *terminal* condition $R(x)$.

Proposition 6 *Let the absolute risk tolerance coefficient* $R(x)$, $x \geq 0$, *be concave (convex). Then, for* $(x,t) \in \mathbb{D}_+$, *the risk tolerance function* $r(x,t)$ *is decreasing (increasing) in* $|\lambda|^2$.

Proof. Let $|\lambda|^2 > |\lambda'|^2$, and denote by $r(x,t;\lambda)$ and $r(x,t;\lambda')$ the corresponding solutions to (22) with $|\lambda|^2, |\lambda'|^2$ being used. We show that, for $(x,t) \in \mathbb{D}_+$, $r(x,t;\lambda) \leq r(x,t;\lambda')$.

As it will be established in Proposition 8, if $R(x)$ is concave, the risk tolerance function $r(x,t)$ is also concave, for each $t \in [0,T)$. Therefore,

$$r_t(x,t;\lambda') + \frac{1}{2}|\lambda|^2 r^2(x,t;\lambda') r_{xx}(x,t;\lambda')$$

$$= r_t(x,t;\lambda') + \frac{1}{2}|\lambda'|^2 r^2(x,t;\lambda') r_{xx}(x,t;\lambda')$$

$$+\frac{1}{2}\left(|\lambda|^2 - |\lambda'|^2\right) r^2(x,t;\lambda') r_{xx}(x,t;\lambda') \leq 0,$$

where we used that $r(x,t;\lambda')$ solves (22) and $r_{xx}(x,t;\lambda') \leq 0$.

Therefore, $r(x,t;\lambda')$ is a super-solution to the equation that $r(x,t;\lambda)$ solves with terminal condition $r(x,T;\lambda) = r(x,T;\lambda')$, and we easily conclude. ■

4 Properties of solutions of the Black's equation

The previous results allow us to derive several properties of the risk tolerance function $r(x,t)$ from analogous properties of the risk tolerance coefficient $R(x)$. Some of these properties have been studied in [8] and [53], but we provide considerably shorter and more direct proofs. Furthermore, we establish various new results.

We show that proving these properties amounts to only specifying the sign of certain nonlinear quantities, which however reduce to much simpler expressions involving the solutions to the heat equations and its spatial derivatives. The latter also solve the heat equation, and thus we can, in turn, use several results for this equation (comparison principle, preservation of log-convexity/concavity, structure of their zero points sets, etc.).

We start with the preservation of strict spatial monotonicity. This question was examined by Arrow [3] who showed that, in a single period problem with one risky stock, the optimal investment in the latter is increasing in wealth if and only if the investor's utility exhibits decreasing absolute risk aversion (*DARA*), and as long as the risk premium is positive. He, also, showed that the fraction of wealth invested in the stock, known as the average propensity to invest, is decreasing in wealth if and only if the utility exhibits increasing relative risk aversion (*IRRA*). Since this seminal work, it has become common in the economic literature to frequently assume that the utility exhibits *DARA* and *IRRA*; these properties are also known as the *Arrow hypothesis.* Similar results were later produced for discrete time models (see, among others, [31], [35], [40], [42], [45] and [46], as well as [15] and [21] and references therein).

Proposition 7 *Let the absolute risk tolerance coefficient* $R(x)$, $x \geq 0$, *be strictly increasing. Then, for each* $t \in [0,T)$, *the risk tolerance function* $r(x,t)$ *is also strictly increasing.*

Proof. Differentiating (16) for $t = T$ yields,

$$R'(x) = \left. \frac{H_{zz}(z,T)}{H_z(z,T)} \right|_{z=H^{(-1)}(x,T)}. \tag{47}$$

Using the monotonicity of $H(z,t)$ and (15), we see that $R(x)$ is strictly increasing if and only if the auxiliary function $H(z,T)$ is strictly concave, $H_{zz}(z,T) > 0$. On the other hand, the function $H_{zz}(z,t)$ also solves the heat equation (13) with terminal condition $H_{zz}(z,T) = \left(I\left(e^{-z}\right)\right)''$. From Proposition 2 and the comparison principle for (13) we deduce that $H_{zz}(z,t) > 0$, $(z,t) \in \mathbb{D}$. Using (21) we have that

$$r_x(x,t) = \left. \frac{H_{zz}(z,t)}{H_z(z,t)} \right|_{z=H^{(-1)}(x,t)},$$

and using that $H_z(x,t) > 0$ we conclude. ■

Next, we examine how the convexity/concavity of risk tolerance coefficient $R(x)$ affects the behavior of the risk tolerance function $r(x,t)$ in *both* space and time.

The curvature of $R(x)$ has been a topic of long-standing debate. We refer the reader to [20] for an extensive discussion as well as to [33] and [38]. As it is argued therein, there are arguments and results which support both assumptions. Among others, it is argued in [24] that a concave risk tolerance coefficient implies that the risk aversion is proper, standard and risk vulnerable (cf., respectively, [43], [32] and [19]). An empirical study in [22] also suggests that the risk tolerance is a concave function of wealth.

For a portfolio problem with explicit solutions with a convex risk tolerance, we refer the reader to [13] and [55].

Proposition 8 *Let the risk tolerance coefficient* $R(x)$, $x \geq 0$, *be concave (convex). Then, the following assertions hold for the risk tolerance function:*

i) for each $x \geq 0$, $r(x,t)$ *is increasing (decreasing) in time,*

ii) for each $t \in [0,T)$, $r(x,t)$ *is concave (convex).*

Proof. We first observe that if (ii) holds, then (i) follows directly from equation (22). To show the former, we argue as follows. Differentiating (21) twice at $t = T$ yields

$$R''(x) = \left. \frac{1}{H_z(z,T)} \frac{\partial^2}{\partial z^2} \left(\log H_z(z,T)\right) \right|_{z=H^{(-1)}(x,T)}. \tag{48}$$

Therefore, $R(x)$ is concave if and only if $H_z(z,T)$ is log-concave. But then, H_z solves the heat equation (13) with positive log-concave terminal data. Applying

Proposition 13 for $h_0(x) = H_z(x,T)$, we deduce that, for each $t \in [0,T)$, the function $H_z(z,t)$ is also log-concave. Differentiating (21) twice yields

$$r_{xx}(x,t) = \frac{1}{H_z(z,t)} \frac{\partial^2}{\partial z^2} (\log H_z(z,t)) \bigg|_{z=H^{(-1)}(x,t)},$$

and using (15) we conclude.

The convex case follows along similar arguments but using, instead, the preservation of the log-convexity property of solutions to the heat equation that $H_z(z,t)$ solves. ■

For the case of concave risk tolerance coefficient, the above result is a bit surprising if the investment horizon $[0,T]$ is long, for it says that the investor should decrease his allocations (in terms of feedback functions) in the risky assets as she gets older. This feature is central in the management of life-cycle funds. We refer the reader, among others, to [7], [10], [47], [50] and [51]. The temporal behavior of the value function and the optimal policies have been examined in more extended model settings in [14] and [34]. Therein, however, the generality of the model did not allow for specific results as the one above.

Next, we assume that the risk tolerance coefficient $R(x)$ is an S-shaped function and examine whether this shape is propagated at previous times. Key role plays the result of [2] for the set of zero points for an auxiliary linear parabolic equation (see (55) below).

Proposition 9 *Assume that there exists a unique $\hat{x} > 0$, such that the risk tolerance coefficient $R(x)$ is convex (concave) in $[0,\hat{x}]$ and concave (convex) in $(\hat{x},\infty)$. Let $G : \mathbb{D} \to \mathbb{R}$ be defined as*

$$G(z,t) := \frac{H_{zzz}(z,t)}{H_{zz}(z,t)} - \frac{H_{zz}(z,t)}{H_z(z,t)}, \tag{49}$$

with H solving (13) and (14), and assume that, for each $t \in [0,T]$,

$$\lim_{z \downarrow -\infty} G(z,t) > 0 \ \ (< 0) \qquad \text{and} \quad \lim_{z \uparrow \infty} G(z,t) < 0 \ (> 0) \tag{50}$$

and

$$G_z(\hat{z},T) \neq 0, \quad \text{for} \ \ \hat{z} = H^{(-1)}(\hat{x},T). \tag{51}$$

Then, there exists a unique continuous curve, $X(t)$, $t \in [0,T]$, with $X(T) = \hat{x}$, such that, for each $t \in [0,T)$, $r(x,t)$ is convex (concave) in $[0, X(t)]$ and concave (convex) in $(X(t),\infty)$.

Proof. We only consider the case that $R(x)$ is convex in $[0,\hat{x}]$ and concave in $(\hat{x},\infty)$, since the other case follows similarly. From (48) we have

$$R''(x) = \frac{H_{zz}(z,T)}{H_z^2(z,T)} G(z,T) \bigg|_{z=H^{(-1)}(x,T)}. \tag{52}$$

Using the assumptions for $R(x)$, the strict monotonicity and full range of $H(x,T)$, we deduce that

$$G(z,T)>0,\ z<\hat{z},\quad G(\hat{z},T)=0,\quad G(z,T)<0,\ z>\hat{z}. \tag{53}$$

Next we show that there exists a unique continuous curve, say $Z(t)$, $t\in[0,T]$, with $Z(T)=\hat{z}$, such that the above properties of $G(z,T)$ are "preserved" at previous times, specifically,

$$G(z,t)>0,\ z<Z(t),\quad G(Z(t),t)=0,\quad G(z,t)<0,\ z>Z(t). \tag{54}$$

For this, define $u(z,t):=\frac{H_{zzz}(z,t)}{H_{zz}(z,t)}$ and $v(z,t):=\frac{H_{zz}(z,t)}{H_{z}(z,t)}$, and observe that $G=u-v$ solves the linear pde

$$G_t(z,t)+\frac{1}{2}G_{zz}(z,t)+u(z,t)\,G_x(z,t)+v_z(z,t)\,G(z,t)=0, \tag{55}$$

with $G(z,T)$ as in (53).

Using the estimates of the partial derivatives of H we easily deduce that G is smooth on $\mathbb{D}$. This together with (51) yield that in a small neighborhood, say $B(\hat{z},t)$, $t\in(T-\varepsilon,T]$, ε small, the ODE $\dot{Z}(t)=-\frac{G_t(Z(t),t)}{G_z(Z(t),t))}$, with $Z(T)=\hat{z}$, has a unique continuous solution $Z(t)$ on which $G(Z(t),t)=0$.

Next, note that assumption (50) implies that for each $t\in[0,T)$, there exists at least one zero point of $G(z,t)$. On the other hand, the results of [2] for the set of zero points for solutions of linear PDE, as the equation (55) above, yield that the set of these points cannot be decreasing as time increases. In other words, the number of zero points of $G(z,t_1)$ is less or equal the number of zero points of $G(z,t_2)$, for $0\le t_1<t_2$. On the other hand, there is a unique curve of zero points in $(T-\varepsilon,T]$, and thus for every $t\in[0,T-\varepsilon]$, there must be also a unique zero point, denoted by $Z(t)$.

To show the continuity of $Z(t)$, $t\in[0,T]$, we argue as follows. Let $t_0\in[0,T)$ and consider a sequence $t_n\to t_0$, with $G(Z(t_n),t_n)=0$. Assumption (50) implies that $Z(t_n)$ is bounded, and therefore along a subsequence, say $Z(t_{k_n})$, we have that $Z(t_{k_n})\to z_0$, for some $z_0\in\mathbb{R}$. Moreover, $G(Z(t_{k_n}),t_{k_n})\to G(z_0,t_0)$ and thus $G(z_0,t_0)=0$. However, for each t_0, there is a unique zero point and, thus, it must be that $z_0=Z(t_0)$. Therefore, there is a unique continuous curve $Z(t)$, $t\in[0,T]$, such that (54) holds.

Recall that, similarly to (52), we have that

$$r_{xx}(x,t)=\left.\frac{H_{zz}(z,t)}{H_z^2(z,t)}G(z,t)\right|_{z=H^{(-1)}(x,t)}, \tag{56}$$

with $\frac{H_{zz}(z,t)}{H_z^2(z,t)}>0$, $(z,t)\in\mathbb{D}$.

Next, we define, for $t\in[0,T]$, the curve

$$X(t):=H(Z(t),t).$$

Let t_0 in $[0,T)$ and $x_0<X(t_0)$. The strict spatial monotonicity of $H^{(-1)}$ yields that $H^{(-1)}(x,t_0)<Z(t_0)$ and, in turn, (54) implies that $G\left(H^{(-1)}(x_0,t_0),t_0\right)>0$. Then, (56) gives $r_{xx}(x_0,t_0)>0$. The rest of the proof follows easily. ■

4.1 The relative risk tolerance

The relative risk tolerance coefficient $\tilde{R}(x)$ and the relative risk tolerance function $\tilde{r}(x,t)$ are defined, for $x>0$, $t\in[0,T]$, as

$$\tilde{R}(x)=-\frac{U'(x)}{xU''(x)} \quad \text{and} \quad \tilde{r}(x,t)=-\frac{u_x(x,t)}{xu_{xx}(x,t)}.$$

There are also given in the implicit form

$$\tilde{R}(H(z,T))=\frac{\partial}{\partial z}\log H(z,T) \quad \text{and} \quad \tilde{r}(H(z,t),t)=\frac{\partial}{\partial z}\log H(z,t), \tag{57}$$

as it follows from (21). It also follows that $\tilde{r}(x,t)$ solves the *Burger's* equation

$$\tilde{r}_t(x,t)+\frac{1}{2}|\lambda|^2\,\tilde{r}_{xx}(x,t)+|\lambda|^2\,\tilde{r}(x,t)\,\tilde{r}_x(x,t)=0,$$

with $\tilde{r}(x,T)=\tilde{R}(x)$.

Representation (57) can be used to prove the following results. The first was proved by [53], and we provide a much shorter proof. The second result is, to the best of our knowledge, new.

Proposition 10 *Assume that the relative risk tolerance coefficient $\tilde{R}(x)$ is increasing (decreasing). Then, for each $t\in(0,T)$, the relative risk tolerance function $\tilde{r}(x,t)$ is also increasing (decreasing).*

Proof. Differentiating (21) at $t=T$ yields, $\tilde{R}'(x)=\frac{\partial^2}{\partial z^2}(\log H(z,T))\Big|_{z=H^{(-1)}(x,T)}$. Therefore, $\tilde{R}(x)$ is an increasing function if and only if $H(z,T)$ is log-convex. Applying Proposition 13 we deduce that, for each $t\in[0,T)$, $H(z,t)$ is log-convex. Using (21) once more, we have that

$$\frac{\partial}{\partial x}(\tilde{r}(x,t))=\frac{\partial^2}{\partial z^2}(\log H(z,t))\Big|_{z=H^{(-1)}(x,t)},$$

and using (15) we conclude.

The analogous results are readily derived when the relative risk tolerance coefficient is decreasing. In this case, one uses the preservation of the log-concavity property of the solution to the heat equation. ■

The following result is related to the concavity/convexity of the relative risk tolerance function.

Proposition 11 *Assume that the function H, solving (13) and (14), satisfies, for $z\in\mathbb{R}$,*

$$(\log H(z,T))_{zz}<0 \quad \textit{and} \quad (\log H(z,T))_{zzz}>0, \tag{58}$$

(resp. $(\log H(z,T))_{zz}>0$ and $(\log H(z,T))_{zzz}<0$). Then, the relative risk tolerance coefficient satisfies $\tilde{R}''(x)>0$ (resp. $\tilde{R}''(x)<0$) and, for $t\in[0,T)$, the relative risk tolerance function is convex (concave) in the spatial variable.

Proof. First, observe that

$$\tilde{R}''(H(z,T)) = \frac{1}{H_z(z,T)} \left(\frac{(\log H(z,T))_{zz}}{H_z(z,T)} \right)_z \tag{59}$$

$$= \frac{1}{H_z(z,T)} \frac{(\log H(z,T))_{zzz} H_z(z,T) - (\log H(z,T))_{zz} H_{zz}(z,T)}{H_z^2(z,T)}.$$

Therefore, if assumption (58) holds, we have that $\tilde{R}''(x) > 0$.

Next, we claim that, for $(z,t) \in \mathbb{D}$, $(\log H(z,t))_{zz} < 0$ and $(\log H(z,t))_{zzz} > 0$.

For simplicity, we take $|\lambda|^2 = 1$. The first assertion is proved in the Appendix. For the second inequality, we introduce the functions $g, F : \mathbb{D} \to \mathbb{R}$, defined as $g(z,t) = \log H(z,t)$ and $F(z,t) = g_{zzz}(z,t)$. Then, F satisfies

$$F_t(z,t) + \frac{1}{2} F_{zz}(z,t) + g_z(z,t) F_z(z,t) + 3 g_{zz}(z,t) F(z,t) = 0,$$

with $F(z,T) > 0$.

Furthermore, $g_{zz}(z,t) < 0$, and $F(z,T) \leq k$, for some positive constant k. The latter follows for direct differentiation and repeated use of (19) and (20). Therefore, applying the comparison principle for the above linear equation yields that $F(z,t) > 0$. Using that

$$\tilde{r}_{xx}(H(z,t),t) = \frac{1}{H_z(z,t)} \frac{F(z,t) H_z(z,t) - g_{zz}(z,t) H_{zz}(z,t)}{H_z^2(z,t)}$$

we obtain that $\tilde{r}_{xx}(H(z,t),t) > 0$ and we conclude.

The second case, $(\log H(z,T))_{zz} > 0$ and $(\log H(z,T))_{zzz} < 0$, is more involved, because the coefficient of the zeroth-order term is now positive, $g_{zz}(z,t) > 0$, as it follows from the assumption that $(\log H(z,T))_{zz} > 0$ and Proposition 13. Note, however, that

$$|(\log H(z,t))_{zz}| = \left| \frac{H_{zzz}(z,t)}{H(z,t)} - \frac{H_{zz}(z,t)}{H(z,t)} \frac{H_z(z,t)}{H(z,t)} \right.$$

$$\left. -2 \frac{H_z(z,t)}{H(z,t)} \left(\frac{H_{zz}(z,t)}{H(z,t)} - \left(\frac{H_z(z,t)}{H(z,t)} \right)^2 \right) \right|,$$

and we conclude using (19) and (20). The rest of the proof follows easily. ■

5 A class of completely monotonic inverse marginals

We consider utility functions whose inverse marginal I is of the form

$$I(x) = \int_a^b x^{-y} \nu(dy), \tag{60}$$

where ν is a positive finite Borel measure, with support $1 < a < b < \infty$. This is a *completely monotonic* function, since $(-1)^n I^{(n+1)}(x) < 0$, $n = 1, 2, ..., x \in \mathbb{R}_+$.

Example: i) $I(x) = x^{-\frac{1}{1-\gamma}}$, $\frac{1}{1-\gamma} \in [a, b]$, which corresponds to the marginal utility $U'(x) = x^{1-\gamma}$.

ii) $I(x) = x^{-\frac{1}{1-\gamma}} + x^{-\frac{2}{1-\gamma}}$, $\frac{1}{1-\gamma} \in \left[a, \frac{b}{2}\right]$, which corresponds to marginal utility $U'(x) = 2^{1-\gamma}\left(\sqrt{1+4x} - 1\right)^{\gamma-1}$.

We then see that (14) yields that $H(x, T)$ is an *absolutely monotonic function, given by*

$$H(z, T) = \int_a^b e^{zy} \nu(dy). \tag{61}$$

It follows easily that, for $n = 1, 2, ...,$

$$\left|xI^{(n)}(x)\right| = x \int_a^b y^n x^{-y-n} \nu(dy) \leq b \int_a^b y^{n-1} x^{-y-(n-1)} \nu(dy) = b \left|I^{(n-1)}(x)\right|,$$

and, turn,

$$a \left|I^{(n-1)}(x)\right| \leq \left|xI^{(n)}(x)\right| \leq b \left|I^{(n-1)}(x)\right|.$$

Furthermore, at $t = T$, $n = 1, 2, ...,$

$$\frac{\partial^n}{\partial z^n} H(z, T) = \int_a^b y^n e^{zy} \nu(dy) > 0, \tag{62}$$

and, thus,

$$a \frac{\partial^{n-1}}{\partial z^{n-1}} H(z, T) \leq \frac{\partial^n}{\partial z^n} H(z, T) \leq b \frac{\partial^{n-1}}{\partial z^{n-1}} H(z, T). \tag{63}$$

The above together with the comparison principle for the heat equation (which follows from (61) and (6)) yield that, for $(z, t) \in \mathbb{D}$,

$$a \frac{\partial^{n-1}}{\partial z^{n-1}} H(z, t) \leq \frac{\partial^n}{\partial z^n} H(z, t) \leq b \frac{\partial^{n-1}}{\partial z^{n-1}} H(z, t). \tag{64}$$

Proposition 12 *Assume that the inverse marginal has the form (60). Then, the following assertions hold:*

i) For $(x, t) \in \mathbb{D}_+$,

$$ax \leq r(x, t) \leq bx.$$

ii) The risk tolerance function is convex in x and decreasing in time, for $(x, t) \in \mathbb{D}_+$.

iii) For $n = 1, 2, ...,$ there exist positive constants K_n, L_n, such that

$$\left|x^{n-1} \frac{\partial^n r(x, t)}{\partial x^n}\right| \leq K_n \quad \text{and} \quad \left|x^n \frac{\partial^n}{\partial x^n}\left(\frac{r(x, t)}{x}\right)\right| \leq L_n. \tag{65}$$

Proof. Part (i) follows directly from (21) and (62).

To show (ii), we first observe that $H_z(z,T)\,H_{zzz}(z,T) - H_{zz}^2(z,T) \geq 0$, as it follows from the inequality

$$\int_a^b y e^{zy} \nu(dy) \int_a^b y^3 e^{zy} \nu(dy) \geq \left(\int_a^b y^2 e^{zy} \nu(dy) \right)^2.$$

Therefore, the function $H_z(z,T)$ is log-convex and we conclude using Propositions 13 and 8.

For part (iii), we first observe that for $i,j = 0,1,2,...$, there exist positive constants C_{ij} such that

$$\left| \frac{\partial}{\partial z} \left(\left(\frac{\partial^j H(z,t)}{\partial z^j} \right)^{-1} \frac{\partial^i H(z,t)}{\partial z^i} \right) \right| \leq C_{ij}, \tag{66}$$

which follows from direct differentiation and repeated application of inequalities (64).

We only establish the first inequality in (65), say for $n=3$, for the rest follows using similar arguments. To this end, observe that we have, where all quantities involving H and its derivatives are evaluated at (z,t),

$$H^2 r_{xxx}(H,t) = \left(\frac{H}{H_z} \right)^2 \left(\left(\frac{H_{zzz}}{H_z} \right)_z - 2\frac{H_{zz}}{H_z} \left(\frac{H_{zz}}{H_z} \right)_z \right.$$
$$\left. -2\frac{H_{zz}}{H_z} \left(\frac{H_{zzz}}{H_z} - \left(\frac{H_{zz}}{H_z} \right)^2 \right) \right),$$

and, using (64) and (66) repeatedly we conclude. ■

We conclude mentioning that because I is a completely monotonic function, Berstein's theorem yields that it can be represented as the Laplace transform for some positive finite measure $\tilde{\nu}$. Indeed, we have

$$I(x) = \int_a^b x^{-y} \nu(dy) = \int_0^\infty e^{-x\rho} \tilde{\nu}(d\rho),$$

with

$$\tilde{\nu}(d\rho) = \left(\int_a^b \frac{\rho^{y-1}}{\Gamma(y)} \nu(dy) \right) d\rho.$$

Complete monotonicity for modeling risk preferences has appeared in [12]. Therein, this structural property is, however, assumed for the marginal utility itself, and not for its inverse. Specifically, the authors consider utility functions with the property

$$U'(x) = \int_0^\infty e^{-xy} \mu(dy), \tag{67}$$

for some finite positive measure μ.

Several interesting question arise. Firstly, what are the utility functions that belong to both classes (60) and (67)? Clearly, power utilities do as well as some combinations of them. However, this is not the case for arbitrary sums of power utilities.

The second question is whether complete monotonicity is being preserved at previous times. Specifically, whether the marginal value function $u_x(x,t)$ remains completely monotonic if the marginal utility $U'(x)$ is as in (67). Similarly, whether $u_x^{(-1)}(x,t)$ remains completely monotonic, if the associated $I(x)$ is as in (60).

Thirdly, it is not clear which utilities from the two classes preserve stochastic dominance of various degrees and, in a different direction, which utilities allow for a dynamic extension (rolling horizon) of the investment problem.

The above questions and other issues related to complete monotonicity of the marginals and their inverses are being currently investigated by the authors in [28].

6 Appendix

We discuss the preservation of the log-convexity and log-concavity of solutions to the heat equation. The log-convexity property is a mere consequence of Hölder's inequality while the log-concavity is more involved. For the latter, we refer the reader to Theorem 1.3 in [11] or to [9] and, for the case of boundary data, to [30]. The one-dimensional case we consider was first proved in [48].

Proposition 13 *Let $h:\mathbb{D}\to\mathbb{R}_+$ be the solution of the heat equation*

$$h_t+\frac{1}{2}\left|\lambda\right|^2 h_{xx}=0,$$

with terminal data $h(x,T)=h_0(x)$, with $h_0\in C^2(\mathbb{R})$ satisfying $h_0(x)>0$ and the growth assumption $h_0(x)\le e^{\gamma x},\gamma>0$. Then, for each $t\in[0,T)$, the following assertions hold.

i) If $h_0(x)$ is a log-convex function, then $h(x,t)$ is also log-convex.

ii) If $h_0(x)$ is a log-concave function, then $h(x,t)$ is also log-concave.

Proof. For simplicity, we set $\left|\lambda\right|^2=1$.

i) We need to show that for $\alpha\in(0,1)$ and $x,y\in\mathbb{R}$,

$$h(\alpha x+(1-\alpha)y,t)\le h(x,t)^{\alpha}\,h(y,t)^{1-\alpha}.$$

The Feynman-Kac formula, the log-convexity of the terminal datum and Hölder's inequality yield

$$\begin{aligned}h(\alpha x+(1-\alpha)y,t)&=E\left(h_0\left(\alpha\left(x+W_{T-t}\right)+(1-\alpha)\left(y+W_{T-t}\right)\right)\right)\\&\le E\left(\left(h_0\left(x+W_{T-t}\right)\right)^{\alpha}\left(h_0\left(y+W_{T-t}\right)\right)^{1-\alpha}\right)\end{aligned}$$

$$\leq \left(E\left(h_0\left(x+W_{T-t}\right)\right)\right)^{\alpha}\left(E\left(h_0\left(y+W_{T-t}\right)\right)\right)^{1-\alpha}$$
$$= \left(h\left(x,t\right)\right)^{\alpha}\left(h\left(y,t\right)\right)^{1-\alpha}.$$

ii) We need to show that for $\alpha \in (0,1)$ and $x, y \in \mathbb{R}$,

$$h\left(\alpha x+(1-\alpha)y,t\right) \geq h\left(x,t\right)^{\alpha} h\left(y,t\right)^{1-\alpha}.$$

The Prékopa-Leindler inequality yields that if, for $0<\alpha<1$, $z, z' \in \mathbb{R}$, positive functions f, m, n satisfy

$$f\left(\alpha z+(1-\alpha)z'\right) \geq \left(m\left(z\right)\right)^{\alpha}\left(n\left(z'\right)\right)^{1-\alpha},$$

then, for $z \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} f\left(z\right)dz \geq \left(\int_{-\infty}^{\infty} m\left(z\right)dz\right)^{\alpha}\left(\int_{-\infty}^{\infty} n\left(z\right)dz\right)^{1-\alpha}.$$

The log-concavity of $h_0\left(x\right)$ yields that for $\alpha \in (0,1)\,,\ z, z' \in \mathbb{R}$,

$$h_0\left(\alpha z+(1-\alpha)z'\right) \geq \left(h_0\left(z\right)\right)^{\alpha}\left(h_0\left(z'\right)\right)^{1-\alpha}.$$

Next, fix $(x,y,t) \in \mathbb{R}\times\mathbb{R}\times[0,T]$ and define the functions

$$f\left(z;x,y,t\right) := e^{-\frac{(\alpha x+(1-\alpha)y-z)^2}{4(T-t)}} h_0\left(z\right)$$

$$m\left(z;x,t\right) := e^{-\frac{(x-z)^2}{4(T-t)}} h_0\left(z\right) \qquad \text{and} \quad n\left(z;y,t\right) := e^{-\frac{(y-z)^2}{4(T-t)}} h_0\left(z\right).$$

We easily see, that

$$f\left(\alpha z+(1-\alpha)z';x,y,t\right) \geq \left(m\left(z;x,t\right)\right)^{\alpha}\left(n\left(z';y,t\right)\right)^{1-\alpha}.$$

Indeed, from the log-concavity of the functions $h_0\left(x\right)$ and e^{-x^2} we have

$$f\left(\alpha z+(1-\alpha)z';x,y,t\right) = e^{-\frac{\left(\alpha x+(1-\alpha)y-\alpha z-(1-\alpha)z'\right)^2}{4(T-t)}} h_0\left(\alpha z+(1-\alpha)z'\right)$$
$$\geq e^{-\frac{\left(\alpha(x-z)+(1-\alpha)(y-z')\right)^2}{4(T-t)}}\left(h_0\left(z\right)\right)^{\alpha}\left(h_0\left(z'\right)\right)^{1-\alpha}$$
$$\geq \left(e^{-\frac{(x-z)^2}{4(T-t)}} h_0\left(z\right)\right)^{\alpha}\left(e^{-\frac{(y-z')^2}{4(T-t)}} h_0\left(z'\right)\right)^{1-\alpha}.$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-\frac{(\alpha x+(1-\alpha)y-z)^2}{4(T-t)}} h_0\left(z\right)dz$$
$$\geq \left(\int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4(T-t)}} h_0\left(z\right)dz\right)^{\alpha}\left(\int_{-\infty}^{\infty} e^{-\frac{(y-z)^2}{4(T-t)}} h_0\left(z\right)dz\right)^{1-\alpha},$$

and we conclude. ■

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