Stochastic Optimization

and

Indifference Valuation

Thaleia Zariphopoulou

UT, Austin
Contents

• Historical perspective

• Beyond derivatives

• Investments and stochastic optimization

• Optimal behavior and valuation

• Integrated pricing systems
Replication
Concept of value

- The option price corresponds to the cost of perfect replication of the option payoff.

- Perfect replication requires strong assumptions about the model.

- In 1973, Black, Scholes and Merton developed such a model.

- Another example was proposed in 1979 by Cox, Ross and Rubinstein.
Does it work in practice?

- In reality no risk can be perfectly replicated.
- The discrepancies depend a lot on the type of risk we look at and on the variables that affect it.
- So why is this idea so fundamental?
- It laid conceptual foundations for the development of a multibillion derivatives industry.
Milestones
Stochastic Calculus and Finance

- In 1979, Harrison and Kreps created a link between pricing by perfect replication and the general theory of martingales.

- The price was shown to correspond to the expected value, calculated under the risk neutral probability, of the discounted option payoff.

- In 1981, Harrison and Pliska presented this stochastic calculus based approach to a large community of researchers from that field.
Term structure models

- Black-Scholes model assumes constant interest rates.

- In reality, the interest rate is not constant and the entire curve of interest rates of different maturities is traded.

- In 1992, Heath, Jarrow and Morton developed a framework for the derivatives pricing under stochastic interest rates.

- At this stage, infinite dimensional analysis and the stochastic PDEs started to play a role in Mathematical Finance.
Volatility models

• Black-Scholes model assumes constant volatility.

• In reality the volatility is not constant and is traded according to a market specific convention.

• It depends on the option strike and maturity.

• This dependence is known as the volatility smile or skew.

• In 1993, Dupire constructed a martingale diffusion that is consistent with the market smile.
Models consistent with smile

- In general, asset evolution is represented by a martingale and the option price may **not** be defined **uniquely** by replication.

- The marginals are given by the market smile/skew — there are **many** ways to construct such a martingale.

- Solutions **differ** by the nature of the **forward** volatilities they generate.

- The notion of value **drifts away** from the concept of payoff replication. The price is defined as the expected value under a martingale measure of the discounted option payoff.
Credit derivatives

- The Black-Scholes model did not have a major impact on the development of this market.

- Is there anything quantitative that the credit markets embraced on their side to the same extent as the other markets accepted the Black-Scholes formula?

- A credit default swap provides information about the distribution of the default time for the underlying name.

- The pricing of tranches of synthetic indices requires the knowledge of the joint distribution of all default times.
Credit derivatives

- The price is defined by an expectation of the payoff, calculated with respect to the joint distribution of the default times.
- The marginals are given by the relevant CDSs.
- The joint distribution is constructed by choosing a copula function.
- The notion of price has nothing to do with the concept of payoff replication.
Beyond derivatives
Funds management

- Funds management industry uses relatively simple mathematics when compared with the derivatives side.

- Mathematical models of investment and consumption are very complex, to an extent more complex than the models used in the derivatives pricing.

- The challenge is to bridge this gap.

- One needs to connect the concept of replication used in the derivatives pricing with the investment optimization.
From replication towards optimal investment

• In a sense one needs to move from: ‘I can do it for you’ – replication, towards: ‘I will do my best’ – optimal investment.

• First steps in the academic literature are being undertaken now.

• Pricing of derivatives has been linked with optimal investment.
Market uncertainty, preferences and investments
Investments and stochastic optimization

• Maximal expected utility models

• Methods

  Primal problem (HJB eqn)

  Dual problem (BS eqn)

• Optimal policies: consumption and portfolios
Maximal expected utility models

- **Market uncertainty**

  \((\Omega, \mathcal{F}, \mathbb{P}), \quad W = (W^1, \ldots, W^d)\)\(\ast\) \(d\)-dim Brownian motion

  Trading horizon: \([0, T], (0, +\infty)\)

  Asset returns: \(dR_t = \mu_t \, dt + \sigma_t \, dW_t\)

  \(\mu \in \mathcal{L}_1(\mathbb{R}^m), \sigma \in \mathcal{L}_2(\mathbb{R}^{d \times m})\)

  riskless asset

  **Wealth process:** \(dX_t = \pi_t \, dR_t - C_t \, dt\)

  **Control processes:** consumption rate \(C_t\), asset allocation \(\pi_t\)
Maximal expected utility models

- **Preferences**: \( U : \mathbb{R} \rightarrow \mathbb{R} \)
  
  increasing, concave, asymptotically elastic....

  \[ U(x) = \frac{1}{\gamma} x^\gamma, \log x, -e^{-\gamma x} \]

- **Objective**: maximize intermediate utility of consumption and utility of terminal wealth

  \[ V(x, t) = \sup_{(C, \pi)} E_P \left( \int_t^T U_1(C_s) \, ds + U_2(X_T)/X_t = x \right) \]

- **Generalizations**: infinite horizon, long-term average, ergodic criteria...
Primal maximal expected utility problem

• $V$ solves the Hamilton-Jacobi-Bellman eqn

$$\begin{cases} 
V_t + F(x, V_x, V_{xx}; U_1) = 0 \\
V(x, T) = U_2(x)
\end{cases}$$

• **Viscosity theory** (Crandall-Lions)

Z., Soner, Touzi, Duffie-Z., Elliott, Davis-Z., Bouchard

• Optimal policies in feedback form

$$C^*_s = \tilde{C}((V^{-1}_x)'(X^*_s, s)) , \quad \pi^*_s = \tilde{\pi}(V_x(X^*_s, s), V_{xx}(X^*_s, s))$$

• Degeneracies, discontinuities, state and control constraints
Dual maximal expected utility problem

- Dual utility functional

\[ U^*(y) = \max_x (U(x) - xy) \]

- Dual problem becomes linear – direct consequence of market completeness and representation, via risk neutrality, of replicable contingent claims

- Problem reduces to an optimal choice of measure – intuitive connection with the so-called state prices

Cox-Huang, Karatzas, Shreve, Cvitanic, Schachermayer, Zitkovic, Kramkov, Delbaen et al, Kabarov, Kallsen, ...
Extensions

- **Recursive utilities and Backward Stochastic Differential Equations (BSDEs)**
  
  Kreps-Porteus, Duffie-Epstein, Duffie-Skiadas, Schroder-Skiadas, Skiadas, El Karoui-Peng-Quenez, Lazrak and Quenez, Hamadene, Ma-Yong, Kobylanski

- **Ambiguity and robust optimization**

  Ellsberg, Chen-Epstein, Epstein-Schneider, Anderson et al., Hansen et al, Maenhout, Uppal-Wang, Skiadas
• Mental accounting and prospect theory
  Discontinuous risk curvature
  Huang-Barberis, Barberis et al., Thaler et al., Gneezy et al.

• Large trader models
  Feedback effects
  Kyle, Platen-Schweizer, Bank-Baum, Frey-Stremme, Back, Cuoco-Cvitanic

• Social interactions
  Continuous of agents – Propagation of fronts
  Malinvaud, Schelling, Glaesser-Scheinkman, Horst-Scheinkman, Foellmer

• Fund management
  Non-zero sum stochastic differential games
  Huggonier-Kaniel
Bridging the gap between optimal behavior and valuation
Pricing elements for contingent claims

“Extreme cases”

- Arbitrage free valuation theory: $\nu(C_T) = E_Q(C_T)$
- Actuarial principles for insurance: $\nu(C_T) = U^{-1}(E_Q(U(C_T)))$

$\mathbb{P}$ historical measure, $\mathbb{Q}$ risk neutral measure,
$U$ utility function

“General case”

- Pricing via equilibrium arguments and investment optimality
Arbitrary risks: $C_T = C(Y_T, S_T)$

Pricing from a perspective of optimal investment

- Use the market to assess both types of risk
- Formulate optimality criteria in terms of individual preferences
- Derive the concept of value from indifference to the various investment opportunities
Issues

- There are hedgeable and unhedgeable risks to specify.
- The hedgeable risks could be priced by arbitrage, and the hedgeable ones by certainty equivalent.
- Two probability measures, the nested risk neutral measure and the historical measure, are then involved.
Indifference valuation

- Optimal behavior with and without the claim

\[ V^0(X_t, t) = \sup_{\pi} E_{\mathbb{P}} \left( \frac{U(X_T)}{\mathcal{F}_t} \right) \]

\[ V^C(X_t, t) = \sup_{\pi} E_{\mathbb{P}} \left( \frac{U(X_T - C_T)}{\mathcal{F}_t} \right) \]

- Partial equilibrium

\[ V^0(X_t, t) = V^C(X_t + \nu_t, t) \]

$\nu_t$: indifference price process

Hodges-Neuberger, Constantinides-Z., Davis, Rouge-El Karoui, Hugonnier et al, Delbaen et all, Musiela-Z., Kramkov-Sirbu, Henderson,...
Structural result

Duality techniques yield

\[ \nu = \sup_Q (E_Q(C_T) - \vartheta(C_T)), \]

\[ \vartheta(C_T) = \frac{1}{\gamma} H(Q/P) - \inf_Q \left( \frac{1}{\gamma} H(Q/P) \right) \]

Rouge–El Karoui, Frittelli, Kabanov and Stricker, Delbaen et al.

Considerations

- Price represented via a non-intuitive optimization problem
- Pricing measure depends on the payoff
- Certain pricing elements are lost
Towards a constitutive analogue of the Black and Scholes theory
Fundamental elements of an indifference pricing system

- Monotonicity, scaling and concavity with respect to payoffs
- Monotonicity, robustness and regularity with respect to risk aversion
- Consistency with the no-arbitrage principle
- Translation invariance with respect to replicable risks
- Risk quantification and monitoring
- Numeraire independence
- Additivity with respect to incremental risks
- Risk transfering across parties
Price representation and payoff decomposition

(Musiela-Z.)

• Complete markets

\[ \nu(C_T, t) = E_Q(C_T/F_t) \]

\[ C_T = \nu(C_T, t) + \int_t^T \frac{\partial \nu(C_T, s)}{\partial S} \, dS_s \]

• Incomplete markets

\[ \nu(C_T, t) = \mathcal{E}_Q(C_T/F_t) \]

\[ C_T = \nu(C_T, t) + \int_t^T \frac{\partial \nu(C_T, s)}{\partial S} \, dS_s \, S_s + R_{t,T} \]

\( R_{t,T} \) : residual risk
Joint work with Marek Musiela (BNP Paribas, London)

**References**


- “Spot and forward dynamic utilities and their associated pricing systems: Case study of the binomial model” Indifference Pricing, PUP (2005)
A new probabilistic representation for indifference prices

The static case

• Arbitrage free prices
  \[ \nu(C_T) = E_{Q^*}(C_T) \]
  
  \[ E(\cdot) : \text{linear pricing functional} \]
  \[ Q^* : \text{the (unique) risk neutral martingale measure} \]

• Indifference prices
  \[ \nu(C_T) = \mathcal{E}_Q(C_T) \]
  
  \[ \mathcal{E} : \text{pricing functional} \]
  \[ Q : \text{pricing measure} \]
  
  \begin{align*}
  \text{nonlinear} & \quad \text{payoff independent} \\
  \text{payoff independent} & \quad \text{preference independent} \\
  \text{wealth independent} & \quad \\
  \text{preference dependent} & \quad 
  \end{align*}
The indifference price

\[ \nu(C_T) = E_Q \left( \frac{1}{\gamma} \log(E_Q(e^{\gamma C(S_T,Y_T) \mid S_T})) \right) = E_Q(C_T) \]

\[ Q(Y_T \mid S_T) = P(Y_T \mid S_T) \]
Choice of pricing measure $\mathbb{Q}$

- $\mathbb{Q}$ needs to be a **martingale measure**
- $\mathbb{Q}$ needs to **preserve the conditional distribution of the unhedgeable risks, given the hedgeable ones, from their historical values**

\[
\mathbb{Q}(Y_T \mid S_T) = \mathbb{P}(Y_T \mid S_T)
\]

**Indifference price components**

\[
\tilde{C}_T = \frac{1}{\gamma} \log \left( E_{\mathbb{P}}(e^{\gamma C(S_T,Y_T)} \mid S_T) \right)
\]

\[
= \frac{1}{\gamma} \log \left( E_{\mathbb{Q}}(e^{\gamma C(S_T,Y_T)} \mid S_T) \right)
\]

\[
\nu(C_T) = E_{\mathbb{Q}}(\tilde{C}_T)
\]
Valuation Procedure

\[ \nu(C_T) = E_Q \left( \frac{1}{\gamma} \log \left( E_Q (e^{\gamma C(S_T, Y_T)} | S_T) \right) \right) = E_Q(C_T) \]

\[ Q(Y_T | S_T) = P(Y_T | S_T) \]

- **Step 1:** Specification, isolation and pricing of unhedgeable risks

The original payoff \( C_T \) is altered to the preference adjusted payoff

\[ \tilde{C}_T = \frac{1}{\gamma} \log \left( E_Q (e^{\gamma C(S_T, Y_T)} | S_T) \right) \]

Observe that

\[ \tilde{C}_T \neq \frac{1}{\gamma} \log \left( E_Q^* (e^{\gamma C(S_T, Y_T)}) \right) \neq \frac{1}{\gamma} \log \left( E_P (e^{\gamma C(S_T, Y_T)}) \right) \]

but

\[ \tilde{C}_T = \frac{1}{\gamma} \log \left( E_Q (e^{\gamma C(S_T, Y_T)} | S_T) \right) = \frac{1}{\gamma} \log \left( E_P (e^{\gamma C(S_T, Y_T)} | S_T) \right) \]

- **Step 2:** Pricing by arbitrage of the remaining hedgeable risks
Payoff decomposition

The static case

Risk quantification and monitoring for unhedgeable risks

- Optimal investments in the presence of the claim

\[ \alpha_{CT,*} = \alpha_{0,*} + \frac{\partial \nu(C_T)}{\partial S_0}, \]

where

\[ \alpha_{0,*} = -\frac{1}{\gamma} \frac{\partial H(Q|\mathbb{P})}{\partial S_0} \]

- Optimal wealths with and without the claim

\[ X_{CT,*}^T = x + \nu(C_T) + \alpha_{CT,*} (S_T - S_0) \quad \text{and} \quad X_{0}^{CT,*} = x + \nu(C_T) \]

\[ X_0^{0,*} = x + \alpha_{0,*} (S_T - S_0) \quad \text{and} \quad X_T^{0,*} = x \]
The optimal residual wealth

\[ L_t = X_t^{CT,*} - X_t^{0,*} \quad \text{for} \quad t = 0, T \]

Indifference hedge and replicability

\[ L_0 = \nu(C_T) \]

and

\[ L_T = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0} (S_T - S_0) \]

Conditional certainty equivalent and extraction of unhedgeable risks

\[ L_T = \tilde{C}_T \]

Martingale property

\[ E_Q(L_T) = L_0 = \nu(C_T) \quad \text{for} \quad Q \in \mathcal{Q}_e \]
• The residual risk $R_t$

\[ R_t = C_t - L_t \quad \text{for} \quad t = 0, T \]

Nonreplicable components of the claim and of the residual risk

\[ R_0 = 0 \quad \text{and} \quad R_T = C_T - \tilde{C}_T ; \quad \tilde{R}_T = 0 \]

The indifference price of the residual risk

\[ \nu(R_T) = 0 \]

Supermartingale property under the pricing measure $\mathbb{Q}$

\[ E_{\mathbb{Q}}(R_T) \leq R_t = 0 \]

Actuarial certainty equivalent

\[ \frac{1}{\gamma} \log E_{\mathbb{P}}(e^{\gamma R_T}) = 0 \]
Residual risk decomposition

The supermartingale $R_t$, for $t = 0, T$, admits the decomposition

$$R_t = R_t^m + R_t^d$$

where

$$R_0^m = 0 \quad \text{and} \quad R_T^m = R_T - E_Q(R_T),$$

and

$$R_0^d = 0 \quad \text{and} \quad R_T^d = E_Q(R_T).$$

The component $R_t^m$ is an $\mathcal{F}_T^{(S,Y)}$-martingale under $Q$ while $R_t^d$ is decreasing and adapted to the trivial filtration $\mathcal{F}_0^{(S,Y)}$. 
Payoff decomposition

Let $\tilde{C}_T$ and $R_T$ be, respectively, the conditional certainty equivalent and the residual risk associated with the claim $C_T$. Let also $R^m_t$ and $R^d_t$ be the Doob decomposition components of the residual risk supermartingale.

Define the process $M_t^{\tilde{C}}$, for $t = 0, T$, by

$$M_0^{\tilde{C}} = \nu(C_T) \quad \text{and} \quad M_T^{\tilde{C}} = \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0)$$

The claim $C_T$ admits the unique, under $\mathbb{Q}$, payoff decomposition

$$C_T = \tilde{C}_T + R_T$$

$$= \nu(C_T) + \frac{\partial \nu(C_T)}{\partial S_0}(S_T - S_0) + R_T = M_T^{\tilde{C}} + R^m_T + R^d_T$$
Convex risk measures $\rho : \mathcal{F}_T \rightarrow \mathcal{R}$

Positions: $C^1_T, C^2_T \in \mathcal{F}_T$

Properties

Convexity: $\rho(\alpha C^1_T + (1 - \alpha)C^2_T) \leq \alpha \rho(C^1_T) + (1 - \alpha)\rho(C^2_T)$

Monotonicity: if $C^1_T \leq C^2_T$ then $\rho(C^1_T) \geq \rho(C^2_T)$

Translation invariance w.r.t. replicable risks: $\rho(C^1_T + m) \leq \rho(C^1_T) - m$

Artzner et al., Foellmer and Shied, Frittelli and Gianini, Delbaen, ...

Convex measures are represented as expectations with penalty terms (entropic terms-special case)
Claim: $C_T \in \mathcal{F}_T$

$$\rho(C_T) = \nu(-C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q \left( e^{-\gamma C_T |S_T} \right) \right)$$

$\rho(C_T)$: Capital requirement in order to accept the position $C_T$

$\nu(C_T)$: Indifference value of the claim $C_T$

The concept of indifference price, deduced from the desire to behave optimally as an investor, is consistent with an axiomatic approach used to determine the capital amount, for a position to be accepted by a supervising body.

- Convex risk measures yield the capital requirement but not the relevant risk monitoring implementation strategy
- Convex risk measures are translation invariant w.r.t. to constant risks
- Convex risk measures are static
Dynamic models and the term structure of risk preferences
The dynamic case

- Specification of the non linear price functional that yields the indifference price representation

\[ \nu_t(C_T) = \mathcal{E}_Q^{(t,T)}(C_T / \mathcal{F}_t) \]

- Semigroup property

\[ \nu_t(C_T) = \mathcal{E}_Q^{(t,t')}(\nu_{t'}(C_T) / \mathcal{F}_t) = \mathcal{E}_Q^{(t,t')}\mathcal{E}_Q^{(t',T)}(C_T / \mathcal{F}_t) \]

- Payoff decomposition

- Residual risk and error quantification

- Proper specification of preferences
A reduced binomial model
The multiperiod model

- **Traded asset**: $S_t$, $t = 0, 1, \ldots, T$ ($S_t > 0, \forall t$)
  \[ \xi_{t+1} = \frac{S_{t+1}}{S_t}, \quad \xi_{t+1} = \xi_d^{t+1}, \xi_u^{t+1} \quad \text{with} \quad 0 < \xi_d^{t+1} < 1 < \xi_u^{t+1} \]

  Second traded asset is riskless yielding zero interest rate

- **Non-traded asset**: $Y_t$, $t = 0, 1, \ldots, T$
  \[ \eta_{t+1} = \frac{Y_{t+1}}{Y_t}, \quad \eta_{t+1} = \eta_d^{t+1}, \eta_u^{t+1} \quad \text{with} \quad \eta_d^t < \eta_u^t \]

  \{S_t, Y_t : t = 0, 1, \ldots, T\} : \text{a two-dimensional stochastic process}

- **Probability space** $(\Omega, (\mathcal{F}_t), \mathbb{P})$
  Filtrations $\mathcal{F}_t^S$ and $\mathcal{F}_t^Y$ : generated by the random variables $S_s$ ($\xi_s$) and $Y_s$ ($\eta_s$), for $s = 0, 1, \ldots, t$. 
**Indifference pricing mechanism**

- **State wealth process:** $X_s, \ s = t + 1, \ldots, T$

  $\alpha_s, \ s = t + 1, t + 2, \ldots, T$: the number of shares of the traded asset held in this portfolio over the time period $[s - 1, s]$

  $$X_T = x + \sum_{s=t+1}^{T} \alpha_s \triangle S_s$$

- **Claim $C_T$** (Path dependence/early exercise are allowed)

- **Value function:** $V^{C_T}(X_t, t; T) = \sup_{\alpha_{t+1}, \ldots, \alpha_T} E_{\mathbb{P}} \left( -e^{-\gamma(X_T - C_T)} \mid \mathcal{F}_t \right)$

- **Indifference price:** $\nu_t(C_T)$

\[
D_{\nu}(X_t, t; T) = V^{C_T}(X_t + \nu_t(C_T), t; T)
\]
Fundamental multiperiod pricing blocks

Let $Z_s$, $0 \leq s \leq t$ be an $\mathcal{F}_s$-adapted process and $Q$ a martingale measure

\[
\mathcal{E}_Q^{(t,s)}(Z_s) \triangleq \mathcal{E}_Q^{(t,s-1)}(\mathcal{E}_Q^{(s-1,s)}(Z_s))
\]

where

\[
\mathcal{E}_Q^{(s-1,s)}(Z_s) \triangleq E_Q \left( \frac{1}{\gamma} \log E_Q(e^{\gamma Z_s} | \mathcal{F}_{s-1} \vee \mathcal{F}_S) | \mathcal{F}_{s-1} \right)
\]

and

\[
\mathcal{E}_Q^{(s,s)}(Z_s) = Z_s
\]

- Note that for $t < s - 1$

\[
\mathcal{E}_Q^{(t,s)}(Z_s) \neq E_Q \left( \frac{1}{\gamma} \log E_Q(Z_s | \mathcal{F}_t \vee \mathcal{F}_S) | \mathcal{F}_t \right)
\]
The valuation algorithm

Let $\mathbb{Q}$ be a martingale measure satisfying, for $t = 0, 1, \ldots, T$

$$\mathbb{Q}(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}_{t+1}^S) = \mathbb{P}(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}_{t+1}^S)$$

• The indifference price $\nu_t(C_T)$ satisfies

$$\nu_t(C_T) = \mathcal{E}_Q^{(t,t+1)}(\nu_{t+1}(C_T)) \quad \nu_T(C_T) = C_T$$

• The indifference price process is given by

$$\nu_t(C_T) = \mathcal{E}_Q^{(t,T)}(C_T)$$

• The pricing algorithm is consistent across time

$$\nu_t(C_T) = \mathcal{E}_Q^{(t,s)}(\mathcal{E}_Q^{(s,T)}(C_T)) = \mathcal{E}_Q^{(t,s)}(\nu_s(C_T)) = \nu_t(\mathcal{E}_Q^{(s,T)}(C_T))$$
Interpretation of the pricing algorithm

Valuation is done via an iterative pricing scheme applied backwards in time.

\[ \nu_t(C_T) = \mathcal{E}^{(t,t+1)}(\mathcal{E}^{(t+1,t+2)}(...(\mathcal{E}^{(T-1,T)}(C_T)))) \]

\[ \nu_t(C_T) = \mathcal{E}^{(t,t+1)}(\nu_{t+1}(C_T)) \]

- Specification, isolation and pricing of unhedgeable risks

\[ \nu_{t+1}(C_T) : \text{the end of the period payoff is altered to the preference adjusted payoff} \]

\[ \tilde{\nu}_{t+1}(C_T) = \frac{1}{\gamma} \log \mathbb{E}_{\mathbb{Q}} \left( e^{\gamma \nu_{t+1}(C_T)} \big| \mathcal{F}_t \vee \mathcal{F}_{t+1}^S \right) \]

- Pricing of remaining hedgeable risks by arbitrage

\[ \nu_t(C_T) = \mathbb{E}_\mathbb{Q}(\tilde{\nu}_{t+1}(C_T)) \]
Indifference prices and quasilinear pdes
Continuous time reduced models

The diffusion case

- **Probabilistic setting**
  - Traded asset: \( S_t, \quad dS_s = \mu(S_s, s)S_s ds + \sigma(S_s, s)S_s dW_s^1 \)
  - Riskless bond \( B_t \)
  - Nontraded asset: \( Y_t, \quad dY_s = b(Y_s, s) ds + a(Y_s, s) dW_s \)
  - Probability space: \((\Omega, \mathcal{F}, (\mathcal{F}_s), \mathbb{P})\)

\[
\mathcal{F}_s = \sigma(W^1_u, W_u; 0 \leq u \leq s) \quad d(W^1, W)(s) = \rho ds
\]

- Claim: \( C_T = C(S_T, Y_T) \)
**Indifference price representation**

The indifference price process is given by

$$\nu_s(C_T) = H(S_s, Y_s, s)$$

where $H(S, y, t)$ is the unique viscosity solution of the **quasilinear** price equation

$$H_t + \mathcal{L}(H(S, y, t)) + \frac{1}{2}\gamma(1 - \rho^2)a^2(y, t)H_y^2 = 0$$

$$H(S, y, T) = C(S, y)$$

The operator $\mathcal{L}(.)$ is given by

$$\mathcal{L}(.) = \frac{1}{2}\sigma^2(S, t)\frac{\vartheta^2}{\partial S^2} + \rho a(y, t)\sigma(S, t)\frac{\vartheta^2}{\partial S\vartheta y} + \frac{1}{2}a^2(S, t)\frac{\vartheta^2}{\vartheta y^2}$$

$$+ \left( b(y, t) - \rho \frac{\mu(S, t)}{\sigma(S, t)}a(y, t) \right) \frac{\vartheta}{\vartheta y}$$
• **Observation 1**

i) $H$ may be directly related to a quadratic stochastic control problem

ii) $H$ may be related to a BSDE

But, there is **no** direct Feynman-Kac type representation of $H$ available up to now!

• **Observation 2**

* Solutions of the above equation are directly structurally related to optimal investment policies in models with stochastic Sharpe ratio, recursive utilities, and stochastic labor income

Stoikov-Z.
Special case

(Musiela-Z., Henderson)

- Lognormal dynamics for the traded asset $S_t$:

$$S_s = \mu S_s ds + \sigma S_s dW_s^1$$

- Payoff depending exclusively on the nontraded asset $Y_t$

$$C(S_T, Y_T) = C(Y_T)$$

$$H(y, t) = \frac{1}{\gamma(1 - \rho^2)} \ln \left( E_Q(e^{\gamma(1-\rho^2)C(Y_T)} | Y_t = y) \right)$$

The above formula is not a naive extension of the classical certainty equivalent price!
Valuation algorithm

- **Pricing measure Q:** a martingale measure satisfying

\[
Q(Y_{t+dt} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t,t+dt}) = P(Y_{t+dt} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t,t+dt})
\]

The indifference price \( \nu_t(S_s, Y_s, s) \) is given by the iterative algorithm

\[
\nu_t(S_s, Y_s, s) = E_Q \left( \frac{1}{\gamma} \log E_Q(e^{\gamma \nu(S_{s+ds}, Y_{s+ds}, s+ds)} \mid \mathcal{F}_s \vee \mathcal{F}^S_{s,s+ds} \mid \mathcal{F}_s) \right)
\]

\[
\nu_T(S_T, Y_T, T) = C(S_T, Y_T)
\]
A novel probabilistic representation of solutions to quasilinear pdes arises

The solution $H(S, y, t)$ of the quasilinear price equation

$$H_t + \mathcal{L}(H(S, y, t)) + \frac{1}{2} \gamma (1 - \rho^2) a^2(y, t) H_y^2 = 0$$

with

$$\mathcal{L}(.) = \frac{1}{2} \sigma^2(S, t) \frac{\vartheta^2}{\vartheta S^2} + \rho a(y, t) \sigma(S, t) \frac{\vartheta^2}{\vartheta S \vartheta y} + \frac{1}{2} a^2(S, t) \frac{\vartheta^2}{\vartheta y^2}$$

$$+ \left( b(y, t) - \rho \frac{\mu(S, t)}{\sigma(S, t)} a(y, t) \right) \frac{\vartheta}{\vartheta y}$$

is given by

$$H(S, y, t) = \lim_{ds \to 0, s \to t} \nu_t(S_{s+ds}, Y_{s+ds}, s + ds)$$

Proof: Based on convergence results of supermartingales, behavior of nonadditive measures and robustness properties of viscosity solutions.
Extensions

• Early exercise claims: \( C_\tau = C(S_\tau, Y_\tau) ; \tau \in \mathcal{F}_{[0,T]} \)

\[
\nu_t^a(C_\tau) = \sup_{\tau \in \mathcal{F}_{[0,T]}} \nu_t(C_\tau) = \sup_{\tau \in \mathcal{F}_{[0,T]}} (\mathcal{E}^{(t,\tau)}(C_\tau))
\]

(Musiela-Z., Oberman-Z., Sokolova)

• Path dependent claims: \( C(t,T) = C(\int_t^T c(S_s, Y_s, s)ds; \omega) \)

• Stochastic interest rates (infinite dimensional problems/PSDEs)
  (Tehranchi and Zariphopoulou)

• Defaultable claims (Bielecki et al.)
Stochastic modelling and indifference prices

- Quasilinear PDEs
- BSDEs
- Stochastic PDEs
- Non-linear expectations
Utility-based pricing systems for general market models
Important issues

- Traditional utilities are “fixed” w.r.t. a given horizon but we need to be able to price claims across all maturities.

- Typically, risk aversion coefficient is chosen in isolation from the market environment.

- Numeraire independence and semigroup property of prices imposes endogeneous restrictions on preferences.
Towards a utility-based pricing system

Ingredients for a term structure of risk preferences that ultimately yields consistent prices across units, numeraires, investment horizons and maturities

- Stochastic risk tolerance

- Time evolution of preferences that captures market incompleteness

This leads to the new notions of

Forward and Spot Dynamic Utilities (Musiela-Z., 2005)
The diffusion case

- Market dynamics

\[ dS_s = \mu(Y_s, s) S_s \, ds + \sigma(Y_s, s) S_s \, dW^1_s \]
\[ dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s \]
\[ \rho = \text{cor}(W^1, W), \quad (\Omega, \mathcal{F}, \mathbb{P}), \quad \lambda_s(Y_s, s) = \frac{\mu(Y_s, s)}{\sigma(Y_s, s)} \]

- Minimal relative entropy measure: \( \mathbb{Q} \)

\[ \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left( - \int_0^T \lambda_s \, dW^1_s - \int_0^T \lambda_1^1 \, dW_{s,1}^{1,\perp} - \frac{1}{2} \int_0^T (\lambda_s^2 + (\lambda_s^\perp)^2) \, ds \right) \]
\[ \lambda^\perp_s = \lambda^\perp_s(Y_s, s); \quad \lambda^\perp(y, t) \sim \text{gradient to the sln of a quasilinear pde} \]

(Hobson, Rheinlander, Stoikov-Z., Benth-Karlsen)
Spot and forward dynamic utilities

• **Forward dynamic utility**

For $T = 1, 2..., \text{ the process } \{U_f(x, t; T) : t = 0, 1, ..., T\}$ defined, for $x \in \mathbb{R}$, by

$$U_f(x, t; T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x} - H_T(Q(\cdot|\mathcal{F}_t)/\mathbb{P}(\cdot|\mathcal{F}_t)) & \text{if } 0 \leq t \leq T - 1
\end{cases}$$

with

$$H_T(Q(\cdot|\mathcal{F}_t)/\mathbb{P}(\cdot|\mathcal{F}_t)) = E_Q \left( \int_t^T \frac{1}{2} (\lambda_s^2 + (\lambda_s^1)^2) \, ds / \mathcal{F}_t \right)$$

is called the **forward to time $T$**, or, normalized at time $T$, **dynamic exponential utility**
• **Spot dynamic utility**

For \( s = 0, 1, \ldots \), the process \( \{U^s(x, t) : t \geq s, \ldots \} \) defined, for \( x \in \mathbb{R} \), by

\[
U^s(x, t; s) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = s \\
-e^{-\gamma x} + \int_s^t \frac{1}{2} \lambda_u^2 \, du & \text{if } t > s
\end{cases}
\]

is called the spot, normalized at time \( s \), dynamic exponential utility.

**Spot and forward dynamic utilities**

• \( U^f \) aggregates information

• \( U^s \) follows information along the path
Spot and forward dynamic value functions

Forward dynamic value function

Let $U^f$ be the forward dynamic utility process, normalized at $T$. The associated forward dynamic value function $V^f$ is defined, for $x \in \mathbb{R}$, $s = 0, 1, \ldots$ and $s \leq t \leq T$, as

$$V^f(x, s, t; T) = \sup_{\pi} E_{\mathbb{P}} \left( U^f(X_t, t; T) \mid \mathcal{F}_t \right)$$

with

$$dX_t = \mu_t \pi_t \, dt + \sigma_t \pi_t \, dW^1_t, \quad t \geq s$$

and $X_s = x$. 
Spot dynamic value function

Let $U^s$ be the spot dynamic utility process, normalized at $s \geq 0$.

The associated spot dynamic value function $V^s$ is defined, for $x \in \mathbb{R}$, and $s \leq t \leq T$ as

$$V^s(x, t, T) = \sup_{\pi} E_{\mathbb{P}} (U^s(X_T, T) | \mathcal{F}_t)$$

with

$$dX_u = \mu_u \pi_u d\mu + \sigma_u \pi_u dW_u^1$$

and $X_t = x$
Fundamental properties of the forward dynamic value function

The forward dynamic value function \( V_f \) has the following properties:

\[
V_f(x, T, T; T) = -e^{-\gamma x} \quad \text{for } x \in \mathbb{R}
\]

\[
V_f(x, s, t_1; T) = V_f(x, s, t_2; T) \quad \text{for } t_1 \neq t_2 \quad \text{and} \quad s \leq \min(t_1, t_2)
\]

\[
V_f(x, s, t; T) = V_f(x, s, s; T) = -e^{-\gamma x - HT(Q(\cdot|F_t)/\mathbb{P}(\cdot|F_t))}
\]

The forward dynamic value function \( V_f \) coincides with the forward dynamic utility

\[
V_f(x, s, t; T) = U_f(x, s; T) \quad \text{for } x \in \mathbb{R} \quad \text{and} \quad s \leq t \leq T
\]
Fundamental properties of the spot dynamic value function

The spot dynamic value function $V^f$ has the following properties:

\[ V^s(x, s, s) = -e^{-\gamma x} \quad \text{for } x \in \mathbb{R} \]

\[ V^s(x, t, T_1) = V^s(x, t, T_2) \quad \text{for } T_1 \neq T_2 \quad \text{and} \quad s \leq t \leq \min(T_1, T_2) \]

\[ V^s(x, t, T) = V^s(x, t, t) = -e^{-\gamma x + \int_s^t \frac{1}{2} \lambda_u^2 \, du} \]

The spot dynamic value function $V^s$ coincides with the spot dynamic utility

\[ V^s(x, t, T) = U^s(x, T) \quad \text{for } x \in \mathbb{R} \quad \text{and} \quad s \leq t \leq T \]
Forward indifference price

- Forward dynamic utility $U^f$, normalized at time $T$

- Let $\bar{T} \leq T$ and consider a claim, written at time $t_0 \geq 0$, yielding payoff $C_{\bar{T}}$ at time $\bar{T}$, with $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$

- Let, also, $V^{f,C_{\bar{T}}}$ and $V^{f,0}$ be, respectively, the forward dynamic value functions with and without the claim

For $t_0 \leq t \leq \bar{T}$, the **forward indifference price**, associated with the forward, to time $T$, utility $U^f$, is defined as the amount $\nu^f(C_{\bar{T}}, t; T)$ for which

$$V^{f,0}(x, t, \bar{T}; T) = V^{f,C_{\bar{T}}}(x + \nu^f(C_{\bar{T}}, t; T), t, \bar{T}; T) \quad \text{for} \quad x \in \mathbb{R}$$
**Spot indifference price**

- Spot dynamic utility $U^s$, normalized at time $s$

- Consider a claim, written at time $t_0 \geq s$, yielding payoff $C_T$ at time $T$, with $C_T \in \mathcal{F}_T$

- Let, also, $V^{s,C_T}$ and $V^{s,0}$ be, respectively, the spot dynamic value functions with and without the claim

For $t_0 \leq t \leq T$, the **spot indifference price**, associated with the spot, normalized at time $s$, utility $U^s$, is defined as the amount $\nu^s(C_T, t; s)$ for which

$$V^{s,0}(x, t, T) = V^{s,C_T}(x + \nu^s(C_T, t), t, T) \quad \text{for} \quad x \in \mathbb{R}$$
Important observation

The spot and forward indifference prices do not in general coincide, i.e., for

\[ s \leq t_0 \leq t \leq \bar{T} \leq T \]

\[ \nu^s(C_{T}, t; s) \neq \nu^f(C_{T}, t; T) \]
Consistency across maturities

Both $V_f$ and $V_s$ are independent of the investment horizon

Spot and forward parity

$$-V_s \left( \int_t^T \frac{\lambda_u^2}{2\gamma} du, t \right) = V_f \left( \int_t^T \frac{1}{2\gamma} du; T \right)$$

$$V_s \left( \int_t^T C_T, t \right) + V_s \left( \int_t^T \frac{1}{2\gamma} du, t \right) = V_f \left( C_T - \int_t^T \frac{1}{2\gamma} du, t; T \right)$$

Spot and forward prices coincide if Sharpe ratios are replicable.
Extensions

• Forward and spot prices in Markovian models
  Connection with non-linear expectations

• Forward and spot prices for stochastic risk tolerance
  The term structure of utilities, stochastic pdes

• Indifference pricing systems and contract theory
  Zero- and Non zero-sum stochastic differential games

• Infinite dimensional models