Indifference valuation

Many questions and few answers

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Towards a constitutive analogue of the Black and Scholes theory in incomplete markets
Modelling, optimal behavior, valuation and risk management

- Market prices of underlying securities
- Risk preference formulation
- Specification of admissible strategies
- Construction of optimal strategies
- Valuation

In complete markets only the first and last step are important
Fundamental elements of an indifference pricing system

Absence of static arbitrage

- Numeraire independence
- Monotonicity, scaling with respect to payoffs
- Monotonicity, robustness with respect to risk aversion
- Translation invariance with respect to replicable risks
- Risk quantification and monitoring
- No indifference value for residual risks
Joint work with Marek Musiela (BNP Paribas, London)

References


• “Spot and forward dynamic utilities and their associated pricing systems: Case study of the binomial model” Indifference Pricing, PUP (2005)

• “Spot indifference valuation in incomplete binomial models” Preprint (2005)

• “Stochastic risk preferences and indifference valuation” Preprint (2005)
Pricing blocks in indifference valuation
A toy incomplete model

- Probability space

\[ \Omega = \{ \omega_1, \omega_2, \omega_3, \omega_4 \}, \quad \mathbb{P} \{ \omega_i \} = p_i, \quad i = 1, \ldots, 4 \]

- Two risks

\[ S_0 \quad \begin{bmatrix} S^u \\ S^d \end{bmatrix} \quad \begin{bmatrix} Y^u \\ Y^d \end{bmatrix} \]

- Random variables \( S_T \) and \( Y_T \)

\[ S_T(\omega_1) = S^u, \quad Y_T(\omega_1) = Y^u \quad S_T(\omega_3) = S^d, \quad Y_T(\omega_3) = Y^u \]
\[ S_T(\omega_2) = S^u, \quad Y_T(\omega_2) = Y^d \quad S_T(\omega_4) = S^d, \quad Y_T(\omega_4) = Y^d \]
Issues

• There are hedgeable and unhedgeable risks to specify

• The hedgeable risks could be priced by arbitrage, and the hedgeable ones by certainty equivalent

• Two probability measures, the nested risk neutral measure and the historical measure, are then involved
Investment opportunities

- We invest the amount $\beta$ in bond ($r = 0$) and the amount $\alpha$ in stock

- Wealth variable

\[ X_0 = x, \quad X_T = \beta + \alpha S_T = x + \alpha (S_T - S_0) \]

Indifference price

- For a general claim $C_T$, we define the value function

\[ V^{C_T}(x) = \max_\alpha E(-e^{-\gamma(X_T - C_T)}) \]

- The indifference price is the amount $\nu(C_T)$ for which,

\[ V^0(x) = V^{C_T}(x + \nu(C_T)) \]
Structural result

Duality techniques yield for general market environments

\[ \nu(C_T) = \sup_Q (E_Q(C_T) - \vartheta), \]

\[ \vartheta = \frac{1}{\gamma} (H(Q|\mathbb{P}) - H(Q|\mathbb{P})) \]

Indifference price representation  (MZ 2004)

- Arbitrage free prices

\[ \nu(C_T) = E_{Q^*}(C_T) \]

\[ E(\cdot) : \text{linear pricing functional} \]
\[ Q^* : \text{the (unique) risk neutral martingale measure} \]

- Indifference prices

\[ \nu(C_T) = E_{\mathcal{E}_Q}(C_T) \]

\[ \mathcal{E} : \text{pricing functional} \]
\[ (\text{possibly}) \text{ nonlinear} \]
\[ \text{payoff independent} \]
\[ \text{wealth independent} \]
\[ \text{preference dependent} \]

\[ Q : \text{pricing measure} \]
\[ \text{payoff independent} \]
\[ \text{preference independent} \]
The indifference price

\[ \nu(C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q(e^{\gamma C(S_T, Y_T)} \mid S_T) \right) = E_Q(C_T) \]

\[ Q(Y_T \mid S_T) = P(Y_T \mid S_T) \]
Static arbitrage
Indifference prices in spot and forward units

**Spot units**

Wealth: \( X^s_T = x + \alpha \left( \frac{S_T}{1+r} - S_0 \right) \)

Value function: \( V^{C_T}(x) = \sup_{\alpha} E^P \left( -e^{-\gamma \left( X^s_T - \frac{C_T}{1+r} \right)} \right) \)

Pricing condition: \( V^0(x) = V^{s,C_T}(x + \nu^s(C_T)) \)

Pricing measure: \( E^{Q^s} \left( \frac{S_T}{1+r} \right) = S_0 \) and \( Q^s(Y_T|S_T) = P(Y_T|S_T) \)

Indifference price: \( \nu^s(C_T) = E^{Q^s} \left( \frac{C_T}{1+r} \right) = E^{Q^s} \left( \frac{1}{\gamma} \log E^{Q^s} \left( e^{\gamma \frac{C_T}{1+r}} | S_T \right) \right) \)
Forward units

Wealth: \[ X_T^f = X_T^s (1 + r) = f + \alpha (F_T - F_0) \; ; \; f = x (1 + r) \]

Value function: \[ V^{CT}(f) = \sup_{\alpha} E_{\mathbb{P}} \left( -e^{-\gamma(X_T^f - C_T)} \right) \]

Pricing condition: \[ V^0(f) = V^{CT}(f + \nu^f(C_T)) \]

Pricing measure: \[ E_{\mathbb{Q}^f}(F_T) = F_0 \; \text{ and } \; Q^f(Y_T|F_T) = \mathbb{P}(Y_T|F_T) \]

Indifference price: \[ \nu^f(C_T) = \mathcal{E}_{\mathbb{Q}^f}(C_T) = E_{\mathbb{Q}^f} \left( \frac{1}{\gamma} \log E_{\mathbb{Q}} \left( e^{\gamma C_T} | F_T \right) \right) \]
Inconsistency across prices expressed in spot and forward units

Pricing measures: \( Q^s = Q^f \)

Spot price: \( \nu^s(C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q \left( e^{\gamma \frac{C_T}{1+r}} | S_T \right) \right) \)

Forward price: \( \nu^f(C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q \left( e^{\gamma C_T} | S_T \right) \right) \)

\[ \nu^f(C_T) \neq (1 + r) \nu^s(C_T) \]
(WWW) **What went wrong?**

- Risk preferences were *not* correctly specified!
- Risk preferences need to be *consistent* across units
- Risk aversion is *not* a constant
Indifference prices in spot and forward units

Spot units

Wealth: \( X_T^s = x + \alpha \left( \frac{S_T}{1+r} - S_0 \right) \)

Value function: \( V^{s,C_T}(x) = \sup_{\alpha} E_{\mathbb{P}} \left( -e^{-\gamma_s(X_T^s - \frac{C_T}{1+r})} \right) \)

Pricing condition: \( V^{s,0}(x) = V^{s,C_T}(x + \nu_s(C_T)) \)

Pricing measure: \( E_{\mathbb{Q}^s} \left( \frac{S_T}{1+r} \right) = S_0 \) and \( \mathbb{Q}^s(Y_T|S_T) = \mathbb{P}(Y_T|S_T) \)

Indifference price: \( \nu_s(C_T) = \mathcal{E}_{\mathbb{Q}^s} \left( \frac{C_T}{1+r} \right) = E_{\mathbb{Q}^s} \left( \frac{1}{\gamma_s} \log E_{\mathbb{Q}^s} \left( e^{\gamma_s \frac{C_T}{1+r}|S_T} \right) \right) \)
Forward units

Wealth: \[ X^f_T = X^s_T (1 + r) = f + \alpha (F_T - F_0); \quad f = x(1 + r) \]

Value function: \[ V^{f,C_T}(f) = \sup_{\alpha} E_{\mathbb{P}} \left( -e^{-\gamma^f (X^f_T - C_T)} \right) \]

Pricing condition: \[ V^{f,0}(f) = V^{f,C_T}(f + \nu^f(C_T)) \]

Pricing measure: \[ E_{Q^f}(F_T) = F_0 \quad \text{and} \quad Q^f(Y_T|F_T) = \mathbb{P}(Y_T|F_T) \]

Indifference price: \[ \nu^f(C_T) = \mathcal{E}_{Q^f}(C_T) = E_{Q^f} \left( \frac{1}{\gamma^f} \log E_{Q^f} \left( e^{\gamma^f C_T|F_T} \right) \right) \]
Consistency across spot and forward units

\[
\nu^f(C_T) = (1 + r)\nu^s(C_T) \iff \delta^s = \frac{1}{1+r}\delta^f
\]

\[
\delta^s = \frac{1}{\gamma^s}, \quad \delta^f = \frac{1}{\gamma^f} : \text{spot and forward risk tolerance}
\]

Risk tolerance is not a number. It is expressed in wealth units.
• Utility functions

\[ U^s(x) = -e^{-\gamma^s x}; \quad x \text{ in spot units} \]

\[ U^f(x) = -e^{-\gamma^f x}; \quad x \text{ in forward units} \]

• Value function representations

\[ V^{s, C_T}(x) = -e^{-\gamma^s (x - \nu^s(C_T))} - H(Q|P) = U^s\left(x - \nu^s(C_T) + \delta^s H(Q|P)\right) \]

\[ V^{f, C_T}(x) = -e^{-\gamma^f (x - \nu^f(C_T))} - H(Q|P) = U^f\left(x - \nu^f(C_T) + \delta^f H(Q|P)\right) \]

\[ Q = Q^s = Q^f \]
Static no arbitrage constraint

Appropriate dependence across units needs to be built into the risk preference structure
The stock as the numeraire

- Indifference price is a unitless quantity (number of stock shares)

- The “utility argument” $\gamma_s T \frac{X_T}{S_T}$ needs to be unitless as well

- Static no arbitrage constraint strongly suggests that risk aversion needs to be stochastic
Stochastic risk preferences
Indifference prices and state dependent risk tolerance

(MZ 2005)

• $\gamma_T = \gamma(S_T)$ $\mathcal{F}_T^S$-measurable random variable
  (in reciprocal to wealth units)

• Risk tolerance (in units of wealth)

  $\delta_T = \frac{1}{\gamma_T}$

• Should $\gamma_T$ be allowed to be $\mathcal{F}_T^{(S,Y)}$-measurable?
Random utility and its value function

- **Value function without the claim**

\[
V^0 (x; \gamma_T) = -\exp \left( -\frac{x}{E_Q \left( \frac{1}{\gamma_T} \right)} - H (Q^* | P) \right)
\]

- **Value function and utility**

\[
V(x, 0; T) = -e^{\frac{x}{E_Q (\frac{1}{\gamma_T})} - H(Q^* | P)} \quad \quad U(X_T; T) = -e^{-\gamma_T X_T}
\]

| 0 | \quad | T |
• Two minimal entropy measures

\[
\frac{dQ^*}{dQ} = \frac{\delta_T}{E_Q(\delta_T)}
\]

\[
E_Q(S_T - (1 + r) S_0) = 0
\]

\[
E_{Q^*}(\gamma_T (S_T - (1 + r) S_0)) = 0
\]

**Structural constraints between the market environment and the risk preferences**
Indifference price and value function

- The indifference price of $C_T$ is given by

$$
\nu (C_T; \gamma_T) = E_Q \left( \frac{1}{\gamma_T} \log E_Q \left( e^{\gamma_T \frac{C_T}{1+r} | S_T} \right) \right)
$$

- The utility

$$
U (X_T; T) = -e^{-\gamma_T X_T}
$$

- Value function with the claim

$$
V^{C_T} (x; \gamma_T) = -\exp \left( - \frac{x - \nu (C_T; \gamma_T)}{E_Q (\delta_T)} \right) - H (Q^* | P)
$$
Optimal policies for stochastic risk preferences in the presence of the claim

\[ \alpha^{C_T,*} = \alpha^{0,*} + \alpha^{1,*} + \alpha^{2,*} \]

- Optimal demand due to market incompleteness: \( \alpha^{0,*} \)

\[ \alpha^{0,*} = -\frac{\partial H(Q^*|\mathbb{P})}{\partial S_0} E_Q(\delta_T) \]
• Optimal demand due to changes in risk tolerance: $\alpha^{1,*}$

$$\alpha^{1,*} = \frac{\partial \log E_Q(\delta_T)}{\partial S_0} x$$

• Optimal demand due to liability: $\alpha^{2,*}$

$$\alpha^{2,*} = E_Q(\delta_T) \frac{\partial}{\partial S_0} \left( \frac{\nu(C_T; \gamma_T)}{E_Q(\delta_T)} \right)$$
Numeraire independence
Indifference prices and general numeraires

The stock as the numeraire

Wealth:

\[ X_{T}^{S} = \frac{x}{S_{T}} + \alpha \left( 1 - \frac{S_{0}}{S_{T}} \right) \]

Value function:

\[ V^{S,CT}(x^{S}) = \sup_{\alpha} E_{\mathbb{P}} \left( e^{-\gamma^{S}(S_{T})(X_{T}^{S} - CT)} \right) \]

Pricing condition:

\[ V^{S,0}(x^{S}) = V^{S,CT}(x^{S} + \nu^{S}(C_{T})) \]

Pricing measure:

\[ Q^{S}(Y_{T}|S_{T}) = \mathbb{P}(Y_{T}|S_{T}) ; \quad \frac{B_{t}}{S_{t}} \text{ martingale w.r.t. } Q^{S} \]
Indifference price

\[ \nu^S(C_T) = E_{QS} \left( \frac{1}{\gamma^S(S_T)} \log E_{QS} \left( e^{\gamma^S(S_T) \frac{C_T}{S_T}} | S_T \right) \right) \]

Numeraire consistency

\[ \frac{\nu(C_T; \gamma_T)}{S_0} = \nu^S(C_T; \gamma^S) \quad \iff \quad \delta_T = \delta^S_T \cdot S_T \]
The term structure of risk preferences
Fundamental questions

• What is the proper specification of the investors’ risk preferences?

• Are risk preferences static or dynamic?

• Are they affected by the market environment and the trading horizon?

• Are there endogenous structural conditions on risk preferences?

• How does the choice of risk preferences affect the indifference prices and the risk monitoring policies?
Requirements for a consistent indifference pricing system

(work in progress MZ)

Risk preferences need to be consistent across units and trading horizons

↓

Dynamic utilities

Martingality of risk tolerance process
Dynamic utilities (MZ 2005)

- Traded asset: $S_t, \quad t \in [0, T]$

- Non-traded asset: $Y_t, \quad t \in [0, T]$

- Filtrations: $\mathcal{F}_t^S, \quad \mathcal{F}_t^Y, \quad \mathcal{F}_t^{(S,Y)}$

- Forward dynamic utility: $U^f(x, t; T)$ is defined as an $\mathcal{F}_t^{(S,Y)}$-measurable process solving

$$U^f(X_s, s : T) = \sup_{\alpha} E_{\mathbb{P}} \left( U^f(X_t, t; T) \middle| \mathcal{F}_s^{(S,Y)} \right)$$

$$X_t = X_s + \int_s^t \alpha_u dS_u$$

$$U^f(x, T; T) = U(x; \omega) \quad \text{given}$$
Spot dynamic utility: $U(x, t)$ is defined as an $\mathcal{F}_t^{(S,Y)}$-measurable process solving

$$U^s(X_s, x) = \sup_{\alpha} E_{\mathbb{P}}(U^s(X_t, t)|\mathcal{F}_s^{(S,Y)})$$

$$X_t = X_s + \int_s^t \alpha_u dS_u$$

$$U^s(x, 0) = U(x; \omega); \quad U \text{ given}$$

Forward dynamic utilities aggregate market information while spot dynamic utilities use information revealed dynamically by the market.
Open questions

• When do forward dynamic utilities exist? Are they robust w.r.t. terminal risk preference specification? (relatively easy)

• When do spot dynamic utilities exist? Are they robust w.r.t. initial risk preference specification? (very hard)

• What structural properties are endogenously imposed on risk tolerance process?

• How is indifference valuation built in terms of forward and spot dynamic preferences?

• Are forward and spot indifference prices equal?
Results to date

Forward and spot dynamic utilities and their associated prices and hedges

- Binomial model
- Diffusion model
- Stochastic risk tolerance

Exponential preferences

Power preferences (partial results)
The multiperiod binomial model

- Traded asset: \( S_t, t = 0, 1, \ldots, T \) (\( S_t > 0, \forall t \))

\[ \xi_{t+1} = \frac{S_{t+1}}{S_t}, \quad \xi_{t+1} = \xi_d \xi_{t+1}, \xi_u \xi_{t+1} \quad \text{with} \quad 0 < \xi_d \xi_{t+1} < 1 < \xi_u \xi_{t+1} \]

Second traded asset is riskless yielding zero interest rate

- Non-traded asset: \( Y_t, t = 0, 1, \ldots, T \)

\[ \eta_{t+1} = \frac{Y_{t+1}}{Y_t}, \quad \eta_{t+1} = \eta_d \eta_{t+1}, \eta_u \eta_{t+1} \quad \text{with} \quad \eta_d < \eta_u \]

\( \{ S_t, Y_t : t = 0, 1, \ldots, T \} \) : a two-dimensional stochastic process

- Probability space \( (\Omega, (\mathcal{F}_t), \mathbb{P}) \)

Filtrations \( \mathcal{F}_t^S \) and \( \mathcal{F}_t^Y \) : generated by the random variables \( S_i (\xi_i) \) and \( Y_i (\eta_i) \), for \( i = 0, 1, \ldots, t. \)
Traditional indifference pricing mechanism

• State wealth process: $X_s$, $s = t + 1, \ldots, T$

$\alpha_s$, $s = t + 1, t + 2, \ldots, T$ : the number of shares of the traded asset held in this portfolio over the time period $[s - 1, s]$

$$X_T = x + \sum_{s=t+1}^{T} \alpha_s \triangle S_s$$

• Claim $C_T$ (Path dependence/early exercise are allowed)

• Value function: $V^{C_T}(X_t, t; T) = \sup_{\alpha_{t+1}, \ldots, \alpha_T} E_{\mathbb{P}} \left( -e^{-\gamma(X_T-C_T)} \mid \mathcal{F}_t \right)$

• Indifference price: $\nu_t(C_T)$

$$V^0(X_t, t; T) = V^{C_T}(X_t + \nu_t(C_T), t; T)$$
Auxiliary quantities quantifying the model incompleteness

- **Local entropy terms:** $h_i$

\[ h_i = q_i \log \frac{q_i}{\mathbb{P}(A_i | \mathcal{F}_{i-1})} + (1 - q_i) \log \frac{1 - q_i}{1 - \mathbb{P}(A_i | \mathcal{F}_{i-1})} \]

with

\[ A_i = \{ \xi_i = \xi_i^u \} \quad \text{and} \quad q_i = \mathbb{Q}(A_i | \mathcal{F}_{i-1}) \]

for $i = 0, 1, \ldots, T$. 
• **Aggregate entropy:** $H_T(Q(\cdot | \mathcal{F}_t) | \mathbb{P}(\cdot | \mathcal{F}_t))$

\[
H_T(Q(\cdot | \mathcal{F}_t) | \mathbb{P}(\cdot | \mathcal{F}_t)) = E_Q \left( \sum_{i=t+1}^{T} h_i | \mathcal{F}_t \right)
\]

\[
h_i = H_i(Q(\cdot | \mathcal{F}_{i-1}) | \mathbb{P}(\cdot | \mathcal{F}_{i-1}))
\]

• **Traditional value function:** $V(x, t)$

\[
V(x, t) = \sup_{\alpha_{t+1}, \ldots, \alpha_T} E_{\mathbb{P}} \left( -e^{-\gamma X_T} | X_t = x \right) = -e^{-\gamma x - H_T}
\]
Forward dynamic utilities (MZ 2005)

For $T = 1, 2..$, the process $\{U^f(x, t; T) : t = 0, 1, ..., T\}$ defined, for $x \in \mathbb{R}$, by

$$U^f(x, t; T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x} - H_T(Q(\cdot | \mathcal{F}_t) \mid \mathbb{P}(\cdot | \mathcal{F}_t)) & \text{if } 0 \leq t \leq T - 1
\end{cases}$$

is the forward to time $T$, or, normalized at time $T$, dynamic exponential utility.
Forward dynamic value function

Let $U^f$ be the forward dynamic utility process, normalized at $T$. The associated forward dynamic value function $V^f$ is defined, for $x \in \mathbb{R}$, $s = 0, 1, ..$ and $s \leq t \leq T$, as

$$V^f(x, s, t; T) = \sup_{\alpha_{s+1}, .., \alpha_t} E_P \left( U^f(X_t, t; T) \mid \mathcal{F}_s \right)$$

with $X_t = X_s + \sum_{i=s+1}^{t} \alpha_i \Delta S_i$ and $X_s = x$. 
Fundamental properties of the forward dynamic value function

The forward dynamic value function $V^f$ has the following properties:

$$V^f(x, T, T; T) = -e^{-\gamma x} \quad \text{for} \quad x \in \mathbb{R}$$

$$V^f(x, s, t_1; T) = V^f(x, s, t_2; T) \quad \text{for} \quad t_1 \neq t_2 \quad \text{and} \quad s \leq \min(t_1, t_2)$$

$$V^f(x, s, t; T) = V^f(x, s, s; T) = -e^{-\gamma x - E_Q\left(\sum_{i=s+1}^{T} h_i | \mathcal{F}_s\right)}$$

The forward dynamic value function $V^f$ coincides with the forward dynamic utility

$$V^f(x, s, t; T) = U^f(x, s; T) \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad s \leq t \leq T$$
Forward indifference price

- Forward dynamic utility $U^f$, normalized at time $T$

- Let $\bar{T} \leq T$ and consider a claim, written at time $t_0 \geq 0$, yielding payoff $C_{\bar{T}}$ at time $\bar{T}$, with $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$

- Let, also, $V^{f,C_{\bar{T}}}$ and $V^{f,0}$ be, respectively, the forward dynamic value functions with and without the claim

For $t_0 \leq t \leq \bar{T}$, the **forward indifference price**, associated with the forward, to time $T$, utility $U^f$, is defined as the amount $\nu^f(C_{\bar{T}}, t; T)$ for which

$$V^{f,0}(x, t, \bar{T}; T) = V^{f,C_{\bar{T}}}(x + \nu^f(C_{\bar{T}}, t; T), t, \bar{T}; T) \quad \text{for} \quad x \in \mathbb{R}$$
Forward dynamic utilities under stochastic risk tolerance

For $T = 1, 2\ldots$, the process $\{U^f(x, t; T) : t = 0, 1, \ldots, T\}$ defined, for $x \in \mathbb{R}$, by

$$U^f(x, t; T) = \begin{cases} 
-e^{-\gamma Tx} & \text{if } t = T \\
-e^{-\frac{x}{\delta_t} - H_T(Q^*(\cdot | \mathcal{F}_t) \mid \mathbb{P}(\cdot | \mathcal{F}_t))} & \text{if } 0 \leq t \leq T - 1
\end{cases}$$

is called the forward to time $T$ dynamic exponential utility with

$$\delta_t = E_Q\left( \frac{1}{\gamma T} \mid \mathcal{F}_t \right) ; \quad \frac{dQ^*}{dQ} = \frac{\delta_T}{E_Q(\delta_T)}$$
Results

• Dynamic value function coincides with dynamic forward utility

• If risk tolerance is consistently specified across units, then prices are numeraire invariant

• Prices are consistent across times, independently of maturities and preference normalization point
Utility-based pricing systems for general market models
The diffusion case

- Market dynamics

\[ dS_s = \mu(Y_s, s)S_s \, ds + \sigma(Y_s, s)S_s \, dW^1_s \]
\[ dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s \]
\[ \rho = \text{cor}(W^1, W), \quad (\Omega, \mathcal{F}, \mathbb{P}), \quad \lambda_s(Y_s, s) = \frac{\mu(Y_s, s)}{\sigma(Y_s, s)} \]

- Minimal relative entropy measure:

\[ \frac{dQ}{dP} = \exp \left( - \int_0^T \lambda_s \, dW^1_s - \int_0^T \lambda_s^1 \, dW^1_{s, \perp} - \frac{1}{2} \int_0^T (\lambda_s^2 + (\lambda_s^\perp)^2) \, ds \right) \]
\[ \lambda_s^\perp = \lambda_s^\perp(Y_s, s); \quad \lambda^\perp(y, t) \sim \text{gradient to the sln of a quasilinear pde} \]

(Hobson, Rheinlander, Stoikov-Z., Benth-Karlsen)
Forward dynamic utilities

For $T > 0$ the process $\{U^f(x, t; T) : 0 \leq t \leq T\}$ defined, for $x \in \mathbb{R}$, by

$$U^f(x, t; T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x} - H_T(Q(\cdot | \mathcal{F}_t) | \mathbb{P}(\cdot | \mathcal{F}_t)) & \text{if } 0 \leq t < T
\end{cases}$$

with

$$H_T(Q(\cdot | \mathcal{F}_t) | \mathbb{P}(\cdot | \mathcal{F}_t)) = E_Q \left( \int_t^T \frac{1}{2} (\lambda_s^2 + (\lambda_s^\perp)^2) \, ds | \mathcal{F}_t \right)$$

is the forward to time $T$, or, normalized at time $T$, dynamic exponential utility.
Open questions

- The risk tolerance must be a martingale

\[ \delta_t = \mathbb{E}_Q \left( \frac{1}{\gamma^T} \mid \mathcal{G}_t \right) \]

- What is \( \mathcal{G}_t \)?

\[ \mathcal{G}_t = \mathcal{F}_t^{(S,Y)} \quad \text{or} \quad \mathcal{G}_t = \mathcal{F}_t^S \ ? \]

- What is the correct measurability of \( \delta_t \)?

\[ \delta_t \in \mathcal{F}_t^S \quad \text{or} \quad \delta_t \in \mathcal{F}_t^{(S,Y)} \ ? \]

- How are the measurability of the stochastic risk tolerance process and the set of admissible policies interrelated?