Indifference pricing systems
and dynamic risk preferences

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Towards a constitutive analogue of the Black and Scholes theory in incomplete markets
Fundamental elements of an indifference pricing system

- Monotonicity, scaling and concavity with respect to payoffs
- Monotonicity, robustness and regularity with respect to risk aversion
- Consistency with the no-arbitrage principle
- Translation invariance with respect to replicable risks
- Risk quantification and monitoring
- Risk transferring across parties
- Additivity with respect to incremental risks
- Numeraire independence
**Fundamental questions**

- What is the proper specification of the investors’ risk preferences?
- Are risk preferences static or dynamic?
- Are they affected by the market environment and the trading horizon?
- Are there endogenous structural conditions on risk preferences?
- How does the choice of risk preferences affect the indifference prices and the risk monitoring policies?
Joint work with Marek Musiela (BNP Paribas, London)

References


- “Spot and forward dynamic utilities and their associated pricing systems: Case study of the binomial model” Indifference Pricing, PUP (2005)

- “Spot indifference valuation in incomplete binomial models” Preprint (2005)
Pricing blocks in indifference valuation
Investment opportunities

• We invest the amount $\beta$ in bond and the amount $\alpha$ in stock

• Wealth variable

$$X_0 = x, \quad X_T = \beta + \alpha S_T = x + \alpha (S_T - S_0)$$

Indifference price

• For a general claim $C_T$, we define the value function

$$V^{C_T}(x) = \max_\alpha E(-e^{-\gamma(X_T - C_T)})$$

• The indifference price is the amount $\nu(C_T)$ for which,

$$V^0(x) = V^{C_T}(x + \nu(C_T))$$
Structural result

Duality techniques yield

\[ \nu = \sup_Q (E_Q(C_T) - \vartheta(C_T)), \]

\[ \vartheta(C_T) = \frac{1}{\gamma} H(Q/P) - \inf_Q \left( \frac{1}{\gamma} H(Q/P) \right) \]


Considerations

• Price represented via a non-intuitive optimization problem
• Pricing measure depends on the payoff
• Certain pricing elements are lost
Indifference price representation \hspace{1cm} (MZ 2004)

- Arbitrage free prices

\[ \nu(C_T) = E_{Q^*}(C_T) \]

\[ E(.) \] : linear pricing functional
\[ Q^* \] : the (unique) risk neutral martingale measure

- Indifference prices

\[ \nu(C_T) = \mathcal{E}_Q(C_T) \]

\[ \mathcal{E} \] : pricing functional
(possibly) nonlinear
payoff independent
wealth independent
preference dependent
\[ Q \] : pricing measure
payoff independent
preference independent
The indifference price

\[ \nu(C_T) = E_\mathbb{Q} \left( \frac{1}{\gamma} \log E_\mathbb{Q}(e^{\gamma C(S_T,Y_T)} \mid S_T) \right) = \mathcal{E}_\mathbb{Q}(C_T) \]

\[ Q(Y_T \mid S_T) = P(Y_T \mid S_T) \]
Valuation Procedure

\[ \nu(C_T) = E_Q \left( \frac{1}{\gamma} \log E_Q(e^{\gamma C(S_T,Y_T)} | S_T) \right) = \mathcal{E}_Q(C_T) \]

\[ Q(Y_T | S_T) = P(Y_T | S_T) \]

- **Step 1:** **Specification, isolation and pricing of unhedgeable risks**

The original payoff \( C_T \) is altered to the **preference adjusted payoff**

\[ \tilde{C}_T = \frac{1}{\gamma} \log E_Q(e^{\gamma C(S_T,Y_T)} | S_T) \]

Observe that

\[ \tilde{C}_T \neq \frac{1}{\gamma} \log E_{Q^*}(e^{\gamma C(S_T,Y_T)} | S_T) \neq \frac{1}{\gamma} \log E_P(e^{\gamma C(S_T,Y_T)} | S_T) \]

but

\[ \tilde{C}_T = \frac{1}{\gamma} \log E_Q(e^{\gamma C(S_T,Y_T)} | S_T) = \frac{1}{\gamma} \log E_P(e^{\gamma C(S_T,Y_T)} | S_T) \]

- **Step 2:** **Pricing by arbitrage of the remaining hedgeable risks**
Numeraire independence
Static no arbitrage constraint

Appropriate dependence across units needs to be built into the risk preference structure
Numeraire independence (MZ PUP-2004)

• Indifference prices in spot and forward units

• Indifference prices and state dependent risk tolerance

• Indifference prices and general numeraires
Indifference prices in spot and forward units

- **Spot units**

  Wealth: \[ X^s_T = x + \alpha \left( \frac{S_T}{1+r} - S_0 \right) \]

  Value function: \[ V^{s,C_T} = \sup_{\alpha} E_{\mathbb{P}} \left( -e^{-\gamma^s(X^s_T - \frac{C_T}{1+r})} \right) \]

  Pricing condition: \[ V^{s,0}(x) = V^{s,C_T}(x + \nu^s(C_T)) \]

  Pricing measure: \[ E_{Q^s} \left( \frac{S_T}{1+r} \right) = S_0 \text{ and } Q^s(Y_T|S_T) = P^s(Y_T|S_T) \]

  Indifference price: \[ \nu^s(C_T) = E_{Q^s} \left( \frac{C_T}{1+r} \right) = E_{Q^s} \left( \frac{1}{\gamma^s} \log E_{Q^s} \left( e^{\gamma^s \frac{C_T}{1+r}} \bigg| S_T \right) \right) \]
• **Forward units**

Wealth: \[ X_T^f = X_T^s (1 + r) = f + \alpha (F_T - F_0); \quad f = x (1 + r) \]

Value function: \[ V^{f,C_T}(f) + \sup_{\alpha} E_P \left( -e^{-\gamma f} (X_T^f - C_T) \right) \]

Pricing condition: \[ V^{f,0}(f) = V^{f,C_T}(f + \nu^f(C_T)) \]

Pricing measure: \[ E_{Q^f}(F_T) = F_0 \quad \text{and} \quad Q^f(Y_T | F_T) = P(Y_T | F_T) \]

Indifference price: \[ \nu^f(C_T) = \mathcal{E}_{Q^f}(C_T) = E_{Q^f} \left( \frac{1}{\gamma f} \log E_{Q^f} \left( e^{\gamma f C_T | F_T} \right) \right) \]
Consistency across spot and forward units

\[ \nu^f(C_T) = (1 + r)\nu^s(C_T) \iff \delta^s = \frac{1}{1+r}\delta^f \]

\[ \delta^s = \frac{1}{\gamma^s}, \quad \delta^f = \frac{1}{\gamma^f} : \text{ spot and forward risk tolerance} \]

Risk tolerance is not a number. It is expressed in wealth units.

- **Value function representations**

\[ V^{s,C_T}(x) = -e^{-\gamma^s(x - \nu^s(C_T))} - H(Q|\mathbb{P}) = U^s \left( x - \nu^s(C_T) + \frac{1}{\delta^s} H(Q|\mathbb{P}) \right) \]

\[ V^{f,C_T}(x) = -e^{-\gamma^f(x - \nu^f(C_T))} - H(Q|\mathbb{P}) = U^f \left( x - \nu^f(C_T) + \frac{1}{\delta^f} H(Q|\mathbb{P}) \right) \]

\[ Q = Q^s = Q^f \]
Indifference prices and state dependent risk tolerance

- $\gamma_T = \gamma(S_T)$ \(\mathcal{F}_T^S\)-measurable random variable
  (in reciprocal to wealth units)

- Risk tolerance (in units of wealth)

$$\delta_T = \frac{1}{\gamma_T}$$

- The indifference price of $C_T$ is given by

$$\nu(C_T; \gamma_T) = E_Q \left( \frac{1}{\gamma_T} \log E_Q \left( e^{\gamma_T \frac{C_T}{1+r} \mid S_T} \right) \right)$$
Value functions and optimal behavior

- Value function without the claim

\[ V^0(x; \gamma_T) = - \exp\left( -\frac{x}{E_Q(\delta_T)} - H(Q^* | P) \right) \]

- Value function with the claim

\[ V^{C_T}(x; \gamma_T) = - \exp\left( -\left( \frac{x - \nu(C_T; \gamma_T)}{E_Q(\delta_T)} \right) - H(Q^* | P) \right) \]
• Two minimal entropy measures

\[
\frac{dQ^*}{dQ}(\omega) = \frac{\delta_T(\omega)}{E_Q(\delta_T)}.
\]

Note: \[ E_Q(S_T - (1 + r)S_0) = 0 \]

\[ E_{Q^*}(\gamma_T(S_T - (1 + r)S_0)) = 0 \]

\[ \downarrow \]

Structural constraints between the market environment and the risk preferences
Optimal policies for stochastic risk preferences in the presence of the claim

\[ \alpha^{CT,*} = \alpha^{0,*} + \alpha^{1,*} + \alpha^{2,*}, \]

- Optimal demand due to incompleteness: \( \alpha^{0,*} \)

\[ \alpha^{0,*} = -\frac{\partial H (Q^* | P)}{\partial S_0} E_Q(\delta_T) \]
• Optimal demand due to changes in risk tolerance: \( \alpha_{1,*} \)

\[
\alpha_{1,*} = \frac{\partial \log E_Q(\delta_T)}{\partial S_0} x
\]

• Optimal demand due to liability: \( \alpha_{2,*} \)

\[
\alpha_{2,*} = E_Q(\delta_T) \frac{\partial}{\partial S_0} \left( \frac{\nu(C_T; \gamma_T)}{E_Q(\delta_T)} \right)
\]
Indifference prices and general numeraires

- The stock as the numeraire

Wealth:

$$X_T^S = \frac{x(1 + r)}{S_T} + \alpha \left( 1 - \frac{S_0}{S_T}(1 + r) \right)$$

Value function:

$$V^{S,C_T} = \sup_{\alpha} E_P \left( -e^{-\gamma^{S}(S_T)(X_T^S - \frac{C_T}{S_T})} \right)$$

Pricing condition:

$$V^{S,0}(x_S) = V^{S,C_T}(x_S + \nu^{S}(C_T))$$

Pricing measure:

$$Q^S(Y_T|S_T) = P(Y_T|S_T) \cdot \frac{B_t}{S_t} \text{ martingale w.r.t. } Q^S$$
Indifference price

\[ \nu^S(C_T) = E_{Q^S} \left( \frac{1}{\gamma^S(S_T)} \log E_{Q^S} \left( e^{\gamma^S(S_T) \frac{C_T}{S_T}} \mid S_T \right) \right) \]

Numeraire consistency

\[ \frac{\nu(C_T; \gamma_T)}{S_0} = \nu^S(C_T; \gamma_T) \quad \iff \quad \delta_T = \delta^S_T \frac{S_T}{1 + r} \]
Spot and forward utilities
Important issues

• Traditional utilities are “fixed” w.r.t. a given horizon but we need to be able to price claims across all maturities.

• Typically, risk aversion coefficient is chosen in isolation from the market environment that changes dynamically.

• Numeraire independence and semigroup property of prices impose endogenous restrictions on preferences.
The multiperiod binomial model

- **Traded asset:** $S_t$, $t = 0, 1, \ldots, T$ ($S_t > 0$, $\forall t$)
  \[ \xi_{t+1} = \frac{S_{t+1}}{S_t}, \quad \xi_{t+1} = \xi_d^{t+1}, \xi_u^{t+1} \quad \text{with} \quad 0 < \xi_d^{t+1} < 1 < \xi_u^{t+1} \]

  Second traded asset is riskless yielding zero interest rate

- **Non-traded asset:** $Y_t$, $t = 0, 1, \ldots, T$
  \[ \eta_{t+1} = \frac{Y_{t+1}}{Y_t}, \quad \eta_{t+1} = \eta_d^{t+1}, \eta_u^{t+1} \quad \text{with} \quad \eta_d^{t} < \eta_u^{t} \]

  \{S_t, Y_t : t = 0, 1, \ldots, T\} : a two-dimensional stochastic process

- **Probability space** $(\Omega, (\mathcal{F}_t), \mathbb{P})$

  Filtrations $\mathcal{F}_t^S$ and $\mathcal{F}_t^Y$ : generated by the random variables $S_i$ ($\xi_i$) and $Y_i$ ($\eta_i$), for $i = 0, 1, \ldots, t$. 
Traditional indifference pricing mechanism

- State wealth process: \( X_s, s = t + 1, ..., T \)

\( \alpha_s, s = t + 1, t + 2, ..., T \) : the number of shares of the traded asset held in this portfolio over the time period \([s - 1, s]\)

\[
X_T = x + \sum_{s=t+1}^{T} \alpha_s \triangle S_s
\]

- Claim \( C_T \) (Path dependence/early exercise are allowed)

- Value function: \( V^{C_T}(X_t, t; T) = \sup_{\alpha_{t+1}, ..., \alpha_T} E_{\mathbb{P}} \left( -e^{-\gamma(X_T-C_T)} \mid \mathcal{F}_t \right) \)

- Indifference price: \( \nu_t(C_T) \)

\[
V^0(X_t, t; T) = V^{C_T}(X_t + \nu_t(C_T), t; T)
\]
Auxiliary quantities quantifying the model incompleteness

- Local entropy terms: $h_i$

\[
h_i = q_i \log \frac{q_i}{\mathbb{P}(A_i | \mathcal{F}_{i-1})} + (1 - q_i) \log \frac{1 - q_i}{1 - \mathbb{P}(A_i | \mathcal{F}_{i-1})}
\]

with

\[
A_i = \{ \xi_i = \xi_i^u \} \quad \text{and} \quad q_i = \mathbb{Q}(A_i | \mathcal{F}_{i-1})
\]

for $i = 0, 1, \ldots, T$. 
• **Aggregate entropy:** \( H_T(Q(\cdot | \mathcal{F}_t) | \mathbb{P}(\cdot | \mathcal{F}_t)) \)

\[
H_T(Q(\cdot | \mathcal{F}_t) | \mathbb{P}(\cdot | \mathcal{F}_t)) = E_Q \left( \sum_{i=t+1}^{T} h_i | \mathcal{F}_t \right)
\]

\[
h_i = H_i(Q(\cdot | \mathcal{F}_{i-1}) | \mathbb{P}(\cdot | \mathcal{F}_{i-1})) .
\]

• **Traditional value function:** \( V(x, t) \)

\[
V(x, t) = \sup_{\alpha_{t+1}, \ldots, \alpha_T} E_\mathbb{P} \left( -e^{-\gamma X_T} / X_t = x \right) = -e^{-\gamma x - H_T}
\]
An example

- Assume $T = 2$ and $U(x; 2) = -e^{-\gamma x}$

- Value function: $V(x, t; 2)$, for $t = 0, 1, 2$.

$$V(x, 0; 2) = -e^{-\gamma x} - E_Q(h_1 + h_2 | \mathcal{F}_0) \quad V(x, 1; 2) = -e^{-\gamma x} - E_Q(h_2 | \mathcal{F}_1)$$

$$V(x, 2; 2) = -e^{-\gamma x}$$

$V(x, 0; 2) \geq V(x, 0; 1)$ and $V(x, 0; 2) > V(x, 0; 1)$, if $\mathbb{P}(A_2 | \mathcal{F}_1) \neq q_2$
Specification of trading horizon?

Time invariant maximal expected utilities: the forward case

- Forward dynamic utility, normalized at $T = 2$

$$U_f^f(x, t; 2) = \begin{cases} 
-e^{-\gamma x} - E_{Q}(h_1+h_2|\mathcal{F}_0) & t = 0 \\
-e^{-\gamma x} - E_{Q}(h_2|\mathcal{F}_1) & t = 1 \\
-e^{-\gamma x} & t = 2 
\end{cases}$$
• Forward dynamic value function $V^f(x, 0, t; 2)$

$$V^f(x, 0, t; 2) = \sup_{\alpha_1, \alpha_t} E_P \left( U^f(X_t, t; 2) \mid \mathcal{F}_t \right) \quad \text{for } t = 1, 2$$

with $X_t = x + \sum_{i=1}^{t} \alpha_i \triangle S_i$.

This choice of dynamic risk preferences makes the associated dynamic value function invariant with respect to the trading horizon

$$V^f(x, 0, 0; 2) = V^f(x, 0, 1; 2) = V^f(x, 0, 2; 2) = -e^{-\gamma x - E_Q(h_1 + h_2 | \mathcal{F}_0)} \quad \text{for } x \in \mathbb{R}$$
Time invariant maximal expected utilities: the spot case

- Spot dynamic utility, normalized at current time 0,

\[
U^s(x, t; 0) = \begin{cases} 
-e^{-\gamma x} & t = 0 \\
-e^{-\gamma x + h_1} & t = 1 \\
-e^{-\gamma x + \sum_{i=1}^{2} h_i} & t = 2 
\end{cases}
\]

- Spot dynamic value function \(V^s(x, 0, t; 0)\)

\[
V^s(x, 0, t; 0) = \sup_{\alpha_1, \alpha_t} E_{\mathbb{P}} (U^s(X_t, t; 0) | \mathcal{F}_0) \quad \text{for } t = 1, 2
\]

with \(X_0 = x\) and \(X_t = x + \sum_{i=1}^{t} \alpha_i \triangle S_i\)
This choice of spot dynamic risk preferences makes the associated spot dynamic value function \textit{invariant} with respect to the trading horizon

\[
V^s(x, 0, 0; 0) = V^s(x, 0, 1; 0) = V^s(x, 0, 2; 0) = -e^{-\gamma x}
\]
Spot and forward dynamic utilities

- **Forward dynamic utility**

For $T = 1, 2, \ldots$, the process $\{U^f(x, t; T) : t = 0, 1, \ldots, T\}$ defined, for $x \in \mathbb{R}$, by

$$U^f(x, t; T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x - H_T(Q(\cdot|\mathcal{F}_t)|\mathbb{P}(\cdot|\mathcal{F}_t))} & \text{if } 0 \leq t \leq T - 1 
\end{cases}$$

is called the forward to time $T$, or, normalized at time $T$, dynamic exponential utility
• **Spot dynamic utility**

For $s = 0, 1, \ldots$, the process $\{U^s(x, t; s) : t = s, s + 1, \ldots\}$ defined, for $x \in \mathbb{R}$, by

$$U^s(x, t; s) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = s \\
-e^{-\gamma x + \sum_{u=s+1}^{t} h_u} & \text{if } t \geq s + 1 
\end{cases}$$

is called the spot, normalized at time $s$, dynamic exponential utility.

**Spot and forward dynamic utilities**

- $U^f(x, t; T) \in \mathcal{F}_t, \ 0 \leq t \leq T$
- $U^s(x, t; s) \in \mathcal{F}_{t-1}, \ 0 \leq s \leq t \leq T$
Spot and forward dynamic value functions

**Forward dynamic value function**

Let $U^f$ be the forward dynamic utility process, normalized at $T$. The associated forward dynamic value function $V^f$ is defined, for $x \in \mathbb{R}$, $s = 0, 1, \ldots$ and $s \leq t \leq T$, as

$$V^f(x, s, t; T) = \sup_{\alpha_{s+1}, \ldots, \alpha_t} E_{\mathbb{P}} \left( U^f(X_t, t; T) | \mathcal{F}_s \right)$$

with $X_t = X_s + \sum_{i=s+1}^{t} \alpha_i \Delta S_i$ and $X_s = x$. 
Spot dynamic value function

Let $U^s$ be the spot dynamic utility process, normalized at $s$, for $s = 0, 1, \ldots$

The associated spot dynamic value function $V^s$ is defined, for $x \in \mathbb{R}$,

$T = s + 1, \ldots \text{ and } s \leq t \leq T$ as

$$V^s(x, t, T; s) = \sup_{\alpha_{t+1}, \ldots, \alpha_T} \mathbb{E}_P (U^s(X_T, T; s) | \mathcal{F}_t)$$

with $X_T = X_t + \sum_{i=t+1}^{T} \alpha_i \Delta S_i$ and $X_t = x$
Fundamental properties of the forward dynamic value function

The forward dynamic value function $V^f$ has the following properties:

$$V^f(x, T, T; T) = -e^{-\gamma x} \text{ for } x \in \mathbb{R}$$

$$V^f(x, s, t_1; T) = V^f(x, s, t_2; T) \text{ for } t_1 \neq t_2 \text{ and } s \leq \min(t_1, t_2)$$

$$V^f(x, s, t; T) = V^f(x, s, s; T) = -e^{-\gamma x - E_Q(\sum_{i=s+1}^{T} h_i|\mathcal{F}_s)}$$

The forward dynamic value function $V^f$ coincides with the forward dynamic utility

$$V^f(x, s, t; T) = U^f(x, s; T) \text{ for } x \in \mathbb{R} \text{ and } s \leq t \leq T$$
**Fundamental properties of the spot dynamic value function** The spot dynamic value function $V^f$ has the following properties:

$$V^s(x, s, s; s) = -e^{-\gamma x} \quad \text{for} \quad x \in \mathbb{R}$$

$$V^s(x, t, T_1; s) = V^s(x, t, T_2; s) \quad \text{for} \quad T_1 \neq T_2 \quad \text{and} \quad s \leq t \leq \min(T_1, T_2)$$

$$V^s(x, t, T; s) = V^s(x, t, t; s) = -e^{-\gamma x + \sum_{i=s+1}^{t} h_i}$$

The spot dynamic value function $V^s$ coincides with the spot dynamic utility

$$V^s(x, t, T; s) = U^s(x, T; s) \quad \text{for} \quad x \in \mathbb{R} \quad \text{and} \quad s \leq t \leq T$$
Forward indifference price

• Forward dynamic utility $U_f$, normalized at time $T$

• Let $\bar{T} \leq T$ and consider a claim, written at time $t_0 \geq 0$, yielding payoff $C_{\bar{T}}$ at time $\bar{T}$, with $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$

• Let, also, $V_f^{C_{\bar{T}}}$ and $V_f^{0}$ be, respectively, the forward dynamic value functions with and without the claim

For $t_0 \leq t \leq \bar{T}$, the forward indifference price, associated with the forward, to time $T$, utility $U_f$, is defined as the amount $\nu^f(C_{\bar{T}}, t; T)$ for which

$$V_f^{0}(x, t, \bar{T}; T) = V_f^{C_{\bar{T}}}(x + \nu^f(C_{\bar{T}}, t; T), t, \bar{T}; T) \quad \text{for} \quad x \in \mathbb{R}$$
**Spot indifference price**

- Spot dynamic utility $U^s$, normalized at time $s$

- Consider a claim, written at time $t_0 \geq s$, yielding payoff $C_T$ at time $T$, with $C_T \in \mathcal{F}_T$

- Let, also, $V^{s,C_T}$ and $V^{s,0}$ be, respectively, the spot dynamic value functions with and without the claim

For $t_0 \leq t \leq T$, the **spot indifference price**, associated with the spot, normalized at time $s$, utility $U^s$, is defined as the amount $\nu^s(C_T, t; s)$ for which

\[
V^{s,0}(x, t, T; s) = V^{s,C_T}(x + \nu^s(C_T, t; s), t, T; s) \quad \text{for} \quad x \in \mathbb{R}
\]
Important observations

• The spot and forward indifference prices do not in general coincide, i.e., for $s \leq t_0 \leq t \leq \bar{T} \leq T$

\[ \nu^s(C_{\bar{T}}, t; s) \neq \nu^f(C_{\bar{T}}, t; T) \]

• The spot indifference prices do not depend on the normalization point, i.e. for $s_1 \neq s_2$, $\max(s_1, s_2) \leq t$,

\[ \nu^s(C_{\bar{T}}, t; s_1) = \nu^s(C_{\bar{T}}, t; s_2) \]
Forward Indifference Prices under Forward Risk Preferences

- **Forward Dynamic Utility**

\[
U^f (x, t; T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x - H_T(Q(\cdot | F_t)| P(\cdot | F_t))} & \text{if } 0 \leq t \leq T - 1
\end{cases}
\]

- **Inverse Forward Dynamic Utility**

\[
(U^f)^{-1} (x, t; T) = -\frac{1}{\gamma} \log (-x) - \frac{1}{\gamma} E_Q \left( \sum_{i=t+1}^{T} h_i | F_t \right)
\]

- **Term structure forward utility notation**

\[
U^f (x, t; T) = U^f_{t,T} (x) \quad \text{and} \quad (U^f)^{-1} (x, t; T) = (U^f)^{-1}_{t,T} (x)
\]
The forward valuation algorithm

Pricing blocks

\( Z \) is a random variable on \((\Omega, \mathcal{F}, \mathbb{P})\) and \(\mathcal{F}_t\) and \(\mathcal{F}_{t}^{S}\) be the market filtrations

- The pricing measure \(\mathbb{Q}\) to be a martingale measure with the minimal relative to \(\mathbb{P}\) entropy, that is, satisfying, for \(t = 0, 1, \ldots\)

\[
\mathbb{Q}\left(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}_{t+1}^{S}\right) = \mathbb{P}\left(\eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}_{t+1}^{S}\right),
\]

- The single step forward price functional

\[
\mathcal{E}_{\mathbb{Q}}^{f,(t,t+1)}(Z; T) = E_{\mathbb{Q}}\left(-\left(U_{t+1}^{f}\right)^{-1} \left(E_{\mathbb{Q}} \left(U_{t+1,T}^{f}(-Z) \mid \mathcal{F}_t \vee \mathcal{F}_{t+1}^{S}\right)\right) \mid \mathcal{F}_t\right)
\]

- The iterative forward price functional

\[
\mathcal{E}_{\mathbb{Q}}^{f,(t,t')} (Z; T) = \mathcal{E}_{\mathbb{Q}}^{f,(t,t+1)} (\mathcal{E}_{\mathbb{Q}}^{f,(t+1,t+2)} (\ldots (\mathcal{E}_{\mathbb{Q}}^{f,(t'-1,t')} (Z; T)); T); T)
\]
The forward valuation algorithm (cont’d)

The forward indifference price \( \nu^f(C_T, t; T) \) is given, for \( t_0 \leq t \leq T \), by the algorithm

\[
\nu^f(C_T, t; T) = \mathcal{E}^{f,(t,t+1)}_{\mathbb{Q}}(\nu^f(C_T, t + 1; T); T), \quad \nu^f(C_T, T; T) = C_T,
\]

The forward indifference price process \( \nu^f(C_T, t; T) \in \mathcal{F}_t \) and satisfies

\[
\nu^f(C_T, t; T) = \mathcal{E}^{f,(t,T)}_{\mathbb{Q}}(C_T; T), \quad t_0 \leq t \leq T
\]

The forward pricing algorithm is consistent across time in that, for \( 0 \leq t_0 \leq t \leq t' \leq T \), the semigroup property

\[
\nu^f(C_T, t; T) = \mathcal{E}^{f,(t,t')}_{\mathbb{Q}}(\mathcal{E}^{f,(t',T)}_{\mathbb{Q}}(C_T; T); T)
\]

\[
= \mathcal{E}^{f,(t,t')}_{\mathbb{Q}}((\nu^f(C_T, t'; T); T) = \nu^f(\mathcal{E}^{f,(t,T)}_{\mathbb{Q}}(C_T; T), t; T)
\]

holds.
Spot Indifference Prices under Spot Risk Preferences

- **Spot Dynamic Utility**

\[
U^s(x, t; s) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = s \\
-e^{-\gamma x + \sum_{u=s+1}^{t} h_u} & \text{if } t \geq s + 1 
\end{cases}
\]

- **Inverse Spot Dynamic Utility**

\[
(U^s)^{-1}(x, t; s) = -\frac{1}{\gamma} \log (-x) - \frac{1}{\gamma} \sum_{i=s+1}^{t} h_i
\]

- **Term structure spot utility notation**

\[
U^s(x, t; s) = U_{s,t}^s(x) \quad \text{and} \quad (U^s)^{-1}(x, t; s) = (U^s)_{s,t}^{-1}(x) \quad \text{for} \quad t \geq s
\]
The spot valuation algorithm

Pricing blocks

$Z$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_t$ and $\mathcal{F}^S_t$ be the market filtrations.

- The pricing measure $\mathbb{Q}$ to be a martingale measure with the minimal relative to $\mathbb{P}$ entropy, that is, satisfying, for $t = 0, 1, \ldots$

$$
\mathbb{Q}\left( \eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t+1} \right) = \mathbb{P}\left( \eta_{t+1} \mid \mathcal{F}_t \vee \mathcal{F}^S_{t+1} \right),
$$

- The single step spot price functional

$$
\mathcal{E}^{s,(t,t+1)}_Q(Z; s) = \mathbb{E}^Q \left( - (U^s)^{-1}_{s,t+1} \left( \mathbb{E}^Q \left( U^s_{s,t+1} (-Z) \mid \mathcal{F}_t \vee \mathcal{F}^S_{t+1} \right) \right) \mid \mathcal{F}_t \right)
$$

- The iterative spot price functional

$$
\mathcal{E}^{s,(t,t')}_Q(Z; s) = \mathcal{E}^{s,(t,t+1)}_Q \left( \mathcal{E}^{s,(t+1,t+2)}_Q \left( \ldots \left( \mathcal{E}^{s,(t'-1,t')}_Q(Z; s) \right) \right); s \right)
$$
The spot valuation algorithm (cont’d)

The spot indifference price $\nu^s(C_T, t; s)$ is given, for $s \leq t \leq T$, by the algorithm

$$\nu^s(C_T, t; s) = \mathcal{E}^s_{Q}^{s,(t,t+1)}(\nu^s(C_T, t + 1; s); s), \quad \nu^s(C_T, T; s) = C_T$$

The spot indifference price process $\nu^s(C_T, t; s) \in \mathcal{F}_t$ and satisfies

$$\nu^s(C_T, t; s) = \mathcal{E}^s_{Q}^{s,(t,T)}(C_T; s), \quad s \leq t_0 \leq t \leq T$$

The spot pricing algorithm is consistent across time in that, for $0 \leq s \leq t_0 \leq t \leq t' \leq T$, the semigroup property

$$\nu^s(C_T, t; s) = \mathcal{E}^s_{Q}^{s,(t,t')} \left( \mathcal{E}^s_{Q}^{s,(t',T)}(C_T; s) ; s \right)$$

$$= \mathcal{E}^s_{Q}^{s,(t,t')} (\nu^s(C_T, t'; s); s) = \nu^s(\mathcal{E}^s_{Q}^{s,(t,T)}(C_T; s), t; s)$$

holds.
When do the forward and spot prices coincide?

**Reduced binomial model**

- Transition probabilities of the traded asset are not affected by the non-traded one

\[ \Pr(\xi_{t+1}^u = \xi_{t+1}^u | \mathcal{F}_t) = \Pr(\xi_{t+1}^u = \xi_{t+1}^u | \mathcal{F}_t^S) \]

The local entropy terms satisfy \( h_t \in \mathcal{F}_t^{S_{t-1}} \), for \( 0 \leq t \leq T \) and thus they create replicable claims

- Then, for all \( C_T \) and \( t \in [s, T] \), the two prices coincide,

\[ \nu_f^f(C_T, t; s) = \nu_s^s(C_T, t; T) \]
Forward and spot prices

- They both capture information about market incompleteness in a dynamic and consistent way.

- The forward price captures information in a coarse, aggregate way. The local entropies are priced as additional non-traded claims. The forward price functional depends functionally on the evolution of the market.

- The spot price captures information in a more dynamic and finer way, along the path. The spot price functional is 'static' in preferences and offers a more intuitive pricing perspective.
Utility-based pricing systems for general market models
The diffusion case

- **Market dynamics**

  \[ dS_s = \mu(Y_s, s)S_s \, ds + \sigma(Y_s, s)S_s \, dW^1_s \]

  \[ dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s \]

  \[ \rho = \text{cor}(W^1, W), \quad (\Omega, \mathcal{F}, \mathbb{P}), \quad \lambda_s(Y_s, s) = \frac{\mu(Y_s, s)}{\sigma(Y_s, s)} \]

- **Minimal relative entropy measure:** \( \mathbb{Q} \)

  \[ \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left( - \int_0^T \lambda_s \, dW^1_s - \int_0^T \lambda_s^1 \, dW^1_{s,\perp} - \frac{1}{2} \int_0^T (\lambda_s^2 + (\lambda_s^\perp)^2) \, ds \right) \]

  \[ \lambda_s^\perp = \lambda_s^\perp(Y_s, s); \quad \lambda^\perp(y, t) \sim \text{gradient to the sln of a quasilinear pde} \]

  (Hobson, Rheinlander, Stoikov-Z., Benth-Karlsen)

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Spot and forward dynamic utilities

- **Forward dynamic utility**

For $T > 0$ the process $\{U^f(x, t; T) : 0 \leq t \leq T\}$ defined, for $x \in \mathbb{R}$, by

$$U^f(x, t; T) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = T \\
-e^{-\gamma x} - H_T(\mathbb{Q}(\cdot|\mathcal{F}_t)/\mathbb{P}(\cdot|\mathcal{F}_t)) & \text{if } 0 \leq t < T
\end{cases}$$

with

$$H_T(\mathbb{Q}(\cdot|\mathcal{F}_t)/\mathbb{P}(\cdot|\mathcal{F}_t)) = E_{\mathbb{Q}} \left( \int_t^T \frac{1}{2} (\lambda_s^2 + (\lambda_s^\perp)^2) \, ds / \mathcal{F}_t \right)$$

is called the forward to time $T$, or, normalized at time $T$, dynamic exponential utility

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• **Spot dynamic utility**

For $s \geq 0$ the process $\{U^s(x, t) : t \geq s\}$ defined, for $x \in \mathbb{R}$, by

$$U^s(x, t; s) = \begin{cases} 
-e^{-\gamma x} & \text{if } t = s \\
-e^{-\gamma x + \int_s^t \frac{1}{2} \lambda^2_u du} & \text{if } t > s 
\end{cases}$$

is called the spot, normalized at time $s$, dynamic exponential utility.

**Spot and forward dynamic utilities**

• $U^f$ aggregates information

• $U^s$ follows and updates information along the path
Spot and forward dynamic value functions

**Forward dynamic value function**

Let $U^f$ be the forward dynamic utility process, normalized at $T$. The associated forward dynamic value function $V^f$ is defined, for $x \in \mathbb{R}$, $s > 0$ and $s \leq t \leq T$, as

$$V^f(x, s, t; T) = \sup_\pi E_\mathbb{P} \left( U^f(X_t, t; T) \mid \mathcal{F}_s \right)$$

with

$$dX_t = \mu_t \pi_t \, dt + \sigma_t \pi_t \, dW_t^1, \quad t \geq s$$

and $X_s = x$. 

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Spot dynamic value function

Let $U^s$ be the spot dynamic utility process, normalized at $s \geq 0$.

The associated spot dynamic value function $V^s$ is defined, for $x \in \mathbb{R}$, and $s \leq t \leq T$ as

$$V^s(x, t, T) = \sup_{\pi} E_{\mathbb{P}} (U^s(X_T, T) \mid \mathcal{F}_t)$$

with

$$dX_u = \mu_u \pi_u du + \sigma_u \pi_u dW_u$$

and $X_t = x$. 
Fundamental properties of the forward dynamic value function

The forward dynamic value function $V^f$ has the following properties:

$$V^f(x, T, T; T) = -e^{-\gamma x} \text{ for } x \in \mathbb{R}$$

$$V^f(x, s, t_1; T) = V^f(x, s, t_2; T) \text{ for } t_1 \neq t_2 \text{ and } s \leq \min(t_1, t_2)$$

$$V^f(x, s, t; T) = V^f(x, s, s; T) = -e^{-\gamma x - H_T(Q(\cdot|\mathcal{F}_s)/P(\cdot|\mathcal{F}_s))}$$

The forward dynamic value function $V^f$ coincides with the forward dynamic utility

$$V^f(x, s, t; T) = U^f(x, s; T) \text{ for } x \in \mathbb{R} \text{ and } s \leq t \leq T$$
Fundamental properties of the spot dynamic value function

The spot dynamic value function $V^f$ has the following properties:

$$V^s (x, s, s) = -e^{-\gamma x} \quad \text{for } x \in \mathbb{R}$$

$$V^s (x, t, T_1) = V^s (x, t, T_2) \quad \text{for } T_1 \neq T_2 \quad \text{and } \ s \leq t \leq \min (T_1, T_2)$$

$$V^s (x, t, T) = V^s (x, t, t) = -e^{-\gamma x} + \int_s^t \frac{1}{2} \lambda_u^2 \, du$$

The spot dynamic value function $V^s$ coincides with the spot dynamic utility

$$V^s (x, t, T) = U^s (x, T) \quad \text{for } x \in \mathbb{R} \quad \text{and } \ s \leq t \leq T$$
Forward indifference price

- Forward dynamic utility $U^f$, normalized at time $T$

- Let $\bar{T} \leq T$ and consider a claim, written at time $t_0 \geq 0$, yielding payoff $C_{\bar{T}}$ at time $\bar{T}$, with $C_{\bar{T}} \in \mathcal{F}_{\bar{T}}$

- Let, also, $V^{f,C_{\bar{T}}}$ and $V^{f,0}$ be, respectively, the forward dynamic value functions with and without the claim

For $t_0 \leq t \leq \bar{T}$, the **forward indifference price**, associated with the forward, to time $T$, utility $U^f$, is defined as the amount $\nu^f(C_{\bar{T}}, t; T)$ for which

$$V^{f,0}(x, t, \bar{T}; T) = V^{f,C_{\bar{T}}}(x + \nu^f(C_{\bar{T}}, t; T), t, \bar{T}; T) \quad \text{for} \quad x \in \mathbb{R}$$
Spot indifference price

- Spot dynamic utility $U^s$, normalized at time $s$

- Consider a claim, written at time $t_0 \geq s$, yielding payoff $C_T$ at time $T$, with $C_T \in \mathcal{F}_T$

- Let, also, $V^{s,C_T}$ and $V^{s,0}$ be, respectively, the spot dynamic value functions with and without the claim

For $t_0 \leq t \leq T$, the **spot indifference price**, associated with the spot, normalized at time $s$, utility $U^s$, is defined as the amount $\nu^s(C_T, t)$ for which

$$V^{s,0}(x, t, T) = V^{s,C_T}(x + \nu^s(C_T, t), t, T) \quad \text{for } x \in \mathbb{R}$$
Important observation

The spot and forward indifference prices do not in general coincide, i.e., for

\[ s \leq t_0 \leq t \leq \bar{T} \leq T \]

\[ \nu^s(C_{\bar{T}}, t; s) \neq \nu^f(C_{\bar{T}}, t; T) \]
Consistency across maturities

Both $V^f$ and $V^s$ are independent of the investment horizon

Spot and forward parity

$$-\nu^s \left( \int_t^T \frac{1}{2} \gamma^2 u \, du, t \right) = \nu^f \left( \int_t^T \frac{1}{2} \gamma^2 u \, du; T \right)$$

$$\nu^s \left( \int_t^T C_T, t \right) + \nu^s \left( \int_t^T -\frac{1}{2} \gamma^2 u \, du, t \right) = \nu^f \left( C_T - \int_t^T \frac{1}{2} \gamma^2 u \, du, t; T \right)$$

Spot and forward prices coincide if Sharpe ratios are replicable.
Extensions

• Forward and spot prices in Markovian models
  Connection with non-linear expectations

• Forward and spot prices for stochastic risk tolerance
  The term structure of utilities, stochastic pdes

• Indifference pricing systems and contract theory
  Zero- and Non zero-sum stochastic differential games

• Infinite dimensional models