Temporal and spatial turnpike-type results under forward time-monotone performance criteria*

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Abstract

We present turnpike-type results for the risk tolerance function in an incomplete market Itô-diffusion setting under time-monotone forward performance criteria. We show that, contrary to the classical case, the temporal and spatial limits do not coincide. Rather, they depend directly on the left- and right-end of the support of an underlying measure associated with the forward performance criterion. We present examples with discrete and continuous such measures, and discuss the asymptotic behavior of the risk tolerance for each case.

1 Introduction

Turnpike results in maximal expected utility models yield the behavior of optimal portfolio functions when the investment horizon is long, under asymptotic assumptions on the investor's risk preferences.

The essence of the "turnpike" result (stated, for simplicity, for a single log-normal stock with coefficients μ and σ) is the following: assume that the investment horizon is $[0,T]$ and that the investor's utility U_T behaves like a power function for large wealth levels, i.e.,

$$U_T(x) \sim \frac{1}{\gamma} x^{\gamma}, \qquad x \quad \text{large}. \tag{1}$$

Then, if this horizon is very long, the associated optimal portfolio function $\pi^*(x,t;T)$ is "close" to the one corresponding to this power utility, i.e., for *each* $x>0$, $t \in [0,T]$,

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$$\frac{\pi^*\left(x,t;T\right)}{x}\sim\frac{\mu}{\sigma^2}\frac{1}{1-\gamma},\qquad T\quad\text{large.}\tag{2}$$

In other words, the asymptotic *spatial* behavior of the terminal datum dictates the long-term *temporal* behavior of the portfolio function for *every* wealth level.

We recall that the function $\pi^*\left(x,t;T\right)$ is the one the determines the optimal wealth process in feedback form, in that the optimal wealth process X_t^*, $t\in\left[0,T\right]$, is generated by the investment strategy $\pi_t^*=\pi^*\left(X_t^*,t;T\right)$.

Turnpike results can be found in [4] where a continuous-time model was first considered, and the turnpike properties were established using contingent claim methods. Their results were later extended in [10] using an autonomous equation that the function $\pi\left(x,t;T\right)$ satisfies and arguments from viscosity solutions. Duality methods were used in [5] for complete markets and the incomplete market case was studied in [9].

More recently, the authors of [2] established the rate of convergence in a log-normal model, showing that there exist a positive constant c and a function $D\left(x\right)$, such that, for all $x>0$,

$$\left|\pi^*\left(x,t;T\right)-\frac{\mu}{\sigma^2}\frac{1}{1-\gamma}x\right|\leq D\left(x\right)e^{-c(T-t)}.$$

A closer look at these turnpike results yields that we are essentially working in a *single* investment horizon setting, $\left[0,T\right]$, which is taken to be very long. As a result, however, one needs to commit to a market model for this long horizon, but this choice cannot be modified later on, if time-consistency is desired. Furthermore, one pre-commits at initial time to a utility function for very far in the future, T. We also remark that no matter how big T is, the optimal investment problem is not defined beyond this point, because the utility function is given for T only.

Herein, we take an alternative point of view. Instead of committing to a single long horizon $[0,T]$, we define an investment problem for all times $t\in[0,\infty)$. Moreover, instead of choosing at an initial time the utility U_T for the remote horizon T, we choose the utility at this initial time. We also depart from the log-normal setting and work with a general Ito-diffusion multi-security incomplete market model.

We measure the performance of investment strategies via the so-called forward investment performance criterion. This criterion was introduced by Musiela and one of the authors in [14] and offers flexibility for performance measurement and risk management under model adaptation and ambiguity, alternative market views, rolling horizons, and others. We recall its definition and refer the reader to, among others, [16], [17], for an overview of the forward approach.

Herein, we focus on the class of time-monotone forward performance criteria, studied in [18] and briefly reviewed in the next section. They are given by a time-decreasing and adapted to the market information process, $U\left(x,t\right)$, $\left(x,t\right)\in\mathbb{R}_+\times\left[0,\infty\right)$, of the form

$$U(x,t) = u(x, A_t),$$

where $u(x,t)$ is a deterministic function (see (14)) and $A_t = \int_0^t |\lambda_s|^2 \, ds$, with the process λ_t being the market price of risk. Note that $U(x,t)$ is a compilation of a deterministic investor-specific input, $u(x,t)$, and a stochastic market-specific input, A_t.

The optimal investment process π_t^* turns out to be, for $t \geq 0$,

$$\pi_t^* = \sigma_t^+ \lambda_t r(X_t^*, A_t) \qquad \text{with} \qquad r(x,t) := -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{3}$$

where σ_t^+ is the pseudo-inverse of the volatility matrix, and X_t^*, $t \geq 0$, the optimal wealth generated by this investment strategy π_t^* (cf. (12)). The function $r(x,t)$ is the (local) risk tolerance and will be the main object of study herein.

Contrary to the classical case, in which a terminal datum is pre-assigned for T and the solution is then constructed for $t \in [0, T)$, in the forward setting, the criterion is defined for all times, starting with an initial (and not terminal) datum $u_0(x) = U(x, 0)$.

In analogy to the classical turnpike setting, we are thus motivated to study the following question: if the initial condition $u_0(x)$ is such that

$$u_0(x) \sim \frac{1}{\gamma} x^\gamma, \qquad x \text{ large}, \tag{4}$$

does this imply that, for each $x > 0$,

$$\frac{r(x,t)}{x} \sim \frac{1}{1-\gamma}, \quad t \text{ large } ?$$

There are fundamental differences between the classical and the forward settings, for one is not a mere variation of the other by a time reversal. Rather, the classical problem is well-posed while the forward is an inverse problem. Naturally, various properties used for the classical turnpike results fail, with the most important being the lack of comparison principle for various PDEs (cf. (14) and (22)) at hand.

The first striking difference between the two settings is the distinct nature of the temporal and spatial limits. Indeed, in the traditional turnpike results in [10] and [2], the temporal limit in (2) coincides with the spatial one, in that for fixed time T_0 and wealth level x_0,

$$\lim_{x \uparrow \infty} \frac{\pi(x, t; T_0)}{x} = \lim_{T \uparrow \infty} \frac{\pi(x_0, t; T)}{x_0}.$$

However, this is *not* the case in the forward setting. Indeed, the temporal and spatial limits of the function $\frac{r(x,t)}{x}$ do *not* coincide. This can be seen, for instance, in the motivational example in section 2.1.

The aim herein then becomes the study of the *spatial* and *temporal* limits

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x} \qquad \text{and} \qquad \lim_{t\uparrow\infty}\frac{r(x_0,t)}{x}, \tag{5}$$

for fixed $t_0 > 0, x_0 > 0$, respectively, under appropriate conditions for the asymptotic behavior of the initial datum $u_0(x)$, for large x.

Pivotal role for determining these limits is played by an underlying positive finite Borel measure, μ, which is the defining element for the construction of the forward performance process. Indeed, it was shown in [18] that the above function u is uniquely (up to an additive constant) related to a harmonic function $h : \mathbb{R}\times[0,\infty) \longrightarrow \mathbb{R}^+$, and, furthermore, the latter is uniquely characterized by an integral transform, specifically,

$$u_x(h(z,t),t) = -e^{-x+\frac{t}{2}} \qquad \text{with} \qquad h(z,t) = \int_a^b e^{zy-\frac{1}{2}y^2t}\mu(dy), \tag{6}$$

for $0 \le a \le b \le \infty$.

An immediate consequence of this general solution is that the initial datum is directly related to this measure μ, in that $(u_0')^{(-1)}$ needs to be of the integral form

$$(u_0')^{(-1)}(x) = \int_a^b x^{-y}\mu(dy).$$

As a result, it is natural to expect that the asymptotic properties of $u_0(x)$, which enter in the turnpike assumptions, are also directly linked to the form and properties of μ.

Furthermore, this measure also appears in the specification of the risk tolerance function. Indeed, we deduce from (3) and (6) that $r(x,t)$ can be represented as

$$r(x,t) = h_x\left(h^{(-1)}(x,t),t\right), \tag{7}$$

with both h_x and $h^{(-1)}$ depending on μ.

The main results herein are that, if the support of the measure is finite, $b < \infty$, then the *spatial* limit coincides with the *right-end* point of the support while the *temporal* limit with the *left-end* one, namely,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x} = b \qquad \text{and} \qquad \lim_{t\uparrow\infty}\frac{r(x_0,t)}{x} = a. \tag{8}$$

The first step in obtaining the above limits is to understand the connection between the asymptotic behavior of the initial (marginal) datum and the finiteness of the measure's support. We study the following two cases, which correspond to the spatial and temporal limits, respectively.

We first show that the asymptotic assumption (4), stated in terms of the marginal,

$$u_0'(x) \quad \sim \quad x^{\gamma-1}, \tag{9}$$

if and only if the right end of the measure's support satisfies both $b = \frac{1}{1-\gamma}$ and $\mu(\{b\}) = 1$. In other words, condition (9) implies that the measure must have finite support with its right boundary equal to $\frac{1}{1-\gamma}$ and, furthermore, with a mass at this point. Conversely, for the measure to have these properties, condition (9) must hold. We then establish the first limit in (8) using representation (6), the equation (14) satisfied by $u(x,t)$, and various convexity properties of h and its derivatives. We stress that the requirement that $\mu(\{b\}) \neq 0$ cannot be relaxed. Indeed, we show in Example 6.2, where the measure is the Lebesgue one, that the spatial turnpike property fails.

For the second case, we relate the finiteness of the measure's support with a relaxed version of (9). We show that if there exists $\gamma < 1$, $\gamma \neq 0$, such that for all $\gamma' \in (\gamma, 1)$ and $\gamma'' < \gamma$,

$$\lim_{x \uparrow \infty} \frac{u_0'(x)}{x^{\gamma'-1}} = 0 \quad \text{and} \quad \lim_{x \uparrow \infty} \frac{u_0'(x)}{x^{\gamma''-1}} = \infty, \tag{10}$$

then the right boundary of the measure's support must satisfy $b = \frac{1}{1-\gamma}$, and vice-versa. This regular variation assumption is weaker than (9), needed for the spatial limit and, naturally, yields a weaker result. Indeed, while the support has to be finite with right boundary equal to $\frac{1}{1-\gamma}$, it does not need to have a mass at $\frac{1}{1-\gamma}$.

We in turn establish the second limit in (8), which is the genuine analogue of the classical turnpike result. Obtaining this limit is considerably more challenging than in the classical case due to the ill-posed nature of the problem. Indeed, the methodology used in [10] is inapplicable because of lack of comparison results for the ergodic version of the equation satisfied by $r(x,t)$. The approach of [2] does not apply either because of the lack of connection between the solutions of the ill-posed heat equation and Feynman-Kac type stochastic representation of its solution. Therefore, one needs to work directly with the function $r(x,t)$, which, from (7) and (6), is given in the implicit form

$$r(x,t) = \int_a^b y e^{y h^{(-1)}(x,t) - \frac{1}{2} y^2 t} \mu(dy),$$

where however the spatial inverse $h^{(-1)}$ is involved.

The key step in obtaining the temporal limit is to show that

$$\lim_{t \uparrow \infty} \frac{h^{(-1)}(x,t)}{t} = \frac{a}{2},$$

where a is the left end point of the measure's support. Then the temporal convergence in (8) and the rate of convergence is shown using the implicit representation

$$r(x,t) - ax = \int_a^b (y-a) e^{ty\left(\frac{h^{(-1)}(x,t)}{t} - \frac{1}{2} y\right)} \mu(dy).$$

In addition to the general spatial and temporal convergence results, we present two representative examples. In the first, the measure is a finite sum of Dirac functions while, in the second, it is taken to be the Lebesgue measure. We calculate the limits of (8), and also provide asymptotic expansions for the risk tolerance function.

The paper is structured as follows. In section 2, we present the market model, the investment performance criterion and a motivating example demonstrating that the temporal and spatial limits do not in general coincide. In sections 3 and 4, we analyze respectively the spatial and temporal asymptotic behavior of the relative risk tolerance, while in section 5 we analyze the asymptotic properties of the relative prudence function. In section 6 we present the two representative examples, and conclude in section 7 with future research directions.

2 The model and the investment performance criterion

The market environment consists of one riskless and k risky securities. The prices of the risky securities are modelled as Itô-diffusion processes, namely, the price S^i of the i^{th} risky asset follows

$$dS_t^i = S_t^i \left(\mu_t^i dt + \Sigma_{j=1}^d \sigma_t^{ji} dW_t^j \right),$$

with $S_0^i > 0$, for $i = 1, ..., k$. The process $W_t = \left(W_t^1, ..., W_t^d \right)$, $t \geq 0$, is a standard Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The coefficients μ_t^i and $\sigma_t^i = \left(\sigma^{1i}, ..., \sigma_t^{di} \right)$, $i = 1, ..., k$, $t \geq 0$, are $\mathcal{F}_t-$adapted processes and values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. We denote by σ_t the volatility matrix, i.e. the $d \times k$ random matrix $\left(\sigma_t^{ji} \right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} asset. We may, then, alternatively, write the above equation as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t \right).$$

The riskless asset, the savings account, has price process B satisfying $dB_t = r_t B_t dt$ with $B_0 = 1$, and for a nonnegative $\mathcal{F}_t-$adapted interest rate process r_t. Also, we denote by μ_t the k-dimensional vector with coordinates μ_t^i and by $\mathbf{1}$ the k-dim vector with every component equal to one. The processes μ_t, σ_t and r_t satisfy the appropriate integrability conditions.

We assume that $\mu_t - r_t \mathbf{1} \in Lin \left(\sigma_t^T \right)$, where $Lin \left(\sigma_t^T \right)$ denotes the linear space generated by the columns of σ_t^T. Therefore, the equation $\sigma_t^T z = \mu_t - r_t \mathbf{1}$ has a solution, known as the market price of risk,

$$\lambda_t = \left(\sigma_t^T \right)^+ \left(\mu_t - r_t \mathbf{1} \right). \tag{11}$$

It is assumed that there exists a deterministic constant $c > 0$, such that $|\lambda_t| \leq c$ and that $\lim_{t \uparrow \infty} \int_0^t |\lambda_s|^2 \, ds = \infty$.

Starting at $t=0$ with an initial endowment $x\geq 0$, the investor invests at any time $t>0$ in the risky and riskless assets. The present value of the amounts invested are denoted by the processes π_t^0 and π_t^i, $i=1,...,k$, respectively, which are taken to be self-financing. The present value of her investment is then given by the discounted wealth process $X_t^\pi=\sum \pi_t^i$, $t>0$, which solves

$$dX_t^\pi = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t) \tag{12}$$

with the (column) vector $\pi_t = \left(\pi_t^i; i=1,...,k\right)$. It is taken to satisfy the non-negativity constraint $X_t^\pi \geq 0$, $t>0$.

The set of admissible policies is given by

$$\mathcal{A}=\left\{\pi : \text{self-financing},\ \ \pi_t\in\mathcal{F}_t,\ E_{\mathbb{P}}\int_0^t |\sigma_s\pi_s|^2\,ds<\infty,\ X_t^\pi\geq 0,\ t>0\right\}.$$

The performance of admissible investment strategies is evaluated via the so-called forward investment performance criteria, introduced in [14] (see, also [15], [16] and [17]). We review their definition next.

We introduce the domain notation $\mathbb{D}_+=\mathbb{R}_+\times[0,\infty)$ and $\mathbb{D}=\mathbb{R}\times[0,\infty)$.

Definition 1 *An $\mathcal{F}_t$-adapted process $U(x,t)$ is a forward investment performance if for $(x,t)\in\mathbb{D}$,*

i) the mapping $x\to U(x,t)$ is strictly increasing and strictly concave;

ii) for each $\pi\in\mathcal{A}$, $E_{\mathbb{P}}(U(X_t^\pi,t))^+<\infty$, and for $s\geq t$,

$$U\left(X_t^\pi,t\right)\geq E_{\mathbb{P}}\left(U(X_s^\pi,s)|\,\mathcal{F}_t\right),$$

iii) there exists $\pi^*\in\mathcal{A}$ such that for $s\geq t$,

$$U\left(X_t^{\pi^*},t\right)=E_{\mathbb{P}}\left(U(X_s^{\pi^*},s)\Big|\,\mathcal{F}_t\right).$$

Herein we focus on the class of *time-monotone* forward performance processes. For the reader's convenience, we rewrite some of the results we stated in the introduction. Time-monotone forward processes were extensively studied in [18], and are given by

$$U(x,t)=u(x,A_t), \tag{13}$$

where $u:\mathbb{D}_+\to\mathbb{R}_+$ is strictly increasing and strictly concave in x, satisfying

$$u_t=\frac{1}{2}\frac{u_x^2}{u_{xx}}. \tag{14}$$

The market input processes A_t and M_t, $t\geq 0$, are defined as

$$M_t=\int_0^t \lambda_s\cdot dW_s \quad \text{ and } \quad A_t=\int_0^t |\lambda_s|^2\,ds=\langle M\rangle_t\,. \tag{15}$$

The optimal portfolio process π_t^* is given by $\pi_t^*=\sigma_t^+\lambda_t r(X_t^*,A_t)$, where the (local) risk tolerance function $r\left(x,t\right):\mathbb{D}_+\to\mathbb{R}_+$ is defined as

$$r\left(x,t\right):=-\frac{u_x\left(x,t\right)}{u_{xx}\left(x,t\right)}. \tag{16}$$

Central role in the construction of the performance criterion, the optimal policies and their wealth plays a harmonic function $h : \mathbb{D} \rightarrow \mathbb{R}_+$, defined via the transformation

$$u_x(h(z,t),t) = e^{-z+\frac{t}{2}}. \tag{17}$$

It solves, as it follows from (14) and (17), the ill-posed heat equation

$$h_t + \frac{1}{2} h_{zz} = 0. \tag{18}$$

Moreover, it is positive and strictly increasing in z. It was shown in [18], that such solutions are *uniquely* represented by

$$h(z,t) = \int_a^b \frac{e^{yz-\frac{1}{2}y^2 t} - 1}{y} \nu(dy) + C,$$

where $a = 0^+$ or $a > 0, b \leq \infty$ and C a generic constant.

The measure ν is defined on $\mathcal{B}^+(\mathbb{R})$, the set of positive Borel measures, with the additional properties that, for $z \in \mathbb{R}$, $\int_a^b e^{yz}\nu(dy) < \infty$ and $\int_a^b \frac{\nu(dy)}{y} < \infty$. To simplify the presentation and without loss of generality, we choose $C := \int_a^b \frac{1}{y}\nu(dy)$ and, also, introduce the normalized measure $\mu\left(dy\right) = \frac{1}{y}\nu(dy)$.

Then, the function h has, for $(z,t) \in \mathbb{D}$, the representation

$$h(z,t) = \int_a^b e^{yz-\frac{1}{2}y^2 t}\mu(dy), \tag{19}$$

with $\int_a^b ye^{yz}\mu(dy) < \infty, \quad a = 0^+,\ a > 0,\ b \leq \infty.$

We easily deduce that for each $t_0 \geq 0$, the function $h\left(.,t_0\right)$ is absolutely monotonic, since $\partial^i h\left(z,t_0\right)/\partial z^i > 0$, $i = 1,2...$ Such functions satisfy, for each $t_0 \geq 0$, $i = 1,2,...$, the inequality

$$\frac{\partial^{i+1} h\left(z,t_0\right)}{\partial z^{i+1}} \frac{\partial^{i-1} h\left(z,t_0\right)}{\partial z^{i-1}} - \left(\frac{\partial^{i} h\left(z,t_0\right)}{\partial z^{i}}\right)^2 > 0. \tag{20}$$

From (17), (16) and (19), we obtain that the risk tolerance function is represented as

$$r(x,t) = h_z\left(h^{(-1)}(x,t),t\right) = \int_a^b ye^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy). \tag{21}$$

Furthermore, the first equality together with (18) yields that it satisfies the (ill-posed) non-linear equation

$$r_t + \frac{1}{2} r^2 r_{xx} = 0, \tag{22}$$

with $r(x,0) = \int_a^b ye^{yh^{(-1)}(x,0)}\mu(dy)$.

We also have that

$$r_x(x,t) = \frac{h_{zz}\left(h^{(-1)}(x,t),t\right)}{r\left(x,t\right)} = \frac{1}{r\left(x,t\right)}\int_a^b y^2 e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2 t}\mu(dy) > 0. \quad (23)$$

Furthermore,

$$r_{xx}\left(x,t\right) = \frac{1}{r^3\left(x,t\right)}\left(h_{zzz}\left(z,t\right)h_z\left(z,t\right) - h_{zz}\left(z,t\right)^2\right)\Big|_{z=h^{(-1)}(x,t)} > 0, \quad (24)$$

where we used (20).

We note that we will frequently differentiate under the integral sign in (19), which is permitted as explained in [18]. It can be also seen directly since, after differentiation, one can show that the relevant integrands are jointly continuous in their respective arguments and thus uniformly locally integrable. This allows us to differentiate under the integral sign (see, for example, Theorem 24.5 in [1] and the remarks following it).

As stated in the introduction, the aim herein is to investigate the spatial and temporal limits in (5), with $r\left(x,t\right)$ as in (21) when the measure has *finite* support. We first provide an example which shows that, contrary to the classical case, these two limits do *not in* general coincide.

2.1 A motivating example

Let the underlying measure μ be a Dirac function at $\frac{1}{1-\gamma}$, $\gamma < 1$. From (19) and (17) we have that, for $t \geq 0$,

$$h(x,t) = e^{\frac{1}{1-\gamma}x - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t} \quad \text{and} \quad u_x(x,t) = x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t}.$$

Therefore, the local risk tolerance function is given by $r(x,t) = \frac{1}{1-\gamma}x$ and thus the spatial and temporal limits coincide,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x} = \frac{1}{1-\gamma} \quad \text{and} \quad \lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0} = \frac{1}{1-\gamma},$$

for fixed t_0, x_0 respectively.

Next, let the measure μ be the sum of two Dirac functions at points $a = \frac{1}{1-\theta}$ and $b = \frac{1}{1-\gamma}$ such that $b = 2a$, with $0 < \theta < 1$ and $\gamma < 1$, i.e.,

$$\mu = \delta_{\frac{1}{1-\theta}} + \delta_{\frac{1}{1-\gamma}} \quad \text{with} \quad \frac{1}{1-\gamma} = 2\frac{1}{1-\theta}. \quad (25)$$

Then, (19) and (17) yield that $h(x,0) = e^{\frac{1}{1-\theta}x} + e^{\frac{1}{1-\gamma}x}$,

$$u_x(x,0) = 2^{1-\theta}\left(\sqrt{1+4x}-1\right)^{\theta-1} \quad \text{and} \quad u_x^{(-1)}(x,0) = x^{-\frac{1}{1-\theta}} + x^{-\frac{1}{1-\gamma}}. \quad (26)$$

In turn,

$$\lim_{x\uparrow\infty}\frac{u_x(x,0)}{x^{\gamma-1}} = \lim_{x\uparrow\infty}\frac{2^{2(1-\gamma)}\left(\sqrt{1+4x}-1\right)^{2(\gamma-1)}}{x^{\gamma-1}} = 1. \quad (27)$$

Moreover, expression (19) gives, for $t > 0$,

$$h(x,t) = e^{\frac{1}{1-\theta}x - \frac{1}{2}\frac{1}{(1-\theta)^2}t} + e^{\frac{2}{1-\theta}x - \frac{1}{2}\frac{2}{(1-\theta)^2}t},$$

and, thus,

$$h^{(-1)}(x,t) = \frac{1}{1-\theta}t + (1-\theta)\ln\left(\frac{\sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t} + 4x} - \sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t}}}{2}\right). \tag{28}$$

In turn, transformation (17) yields

$$u_x(x,t) = 2^{1-\theta} e^{\left(\frac{1}{2} - \frac{1}{1-\theta}\right)t}\left(\sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t} + 4x} - \sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t}}\right)^{\gamma-1}.$$

Differentiating the above to obtain $u_{xx}(x,t)$ (or using (19), (28) and (21)), we deduce that the risk tolerance function is given by

$$r(x,t) = \frac{x}{1-\gamma}\frac{\sqrt{4x + e^{\left(\frac{1}{1-\theta}\right)^2 t}}}{\sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t} + 4x} + \sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t}}}. \tag{29}$$

Therefore, for each $t_0 \geq 0$,

$$\lim_{x\uparrow\infty}\frac{r(x,t_0)}{x} = \frac{2}{1-\theta} = \frac{1}{1-\gamma}. \tag{30}$$

while, for each $x_0 > 0$,

$$\lim_{t\uparrow\infty}\frac{r(x_0,t)}{x_0} = \frac{1}{1-\theta}. \tag{31}$$

Therefore, the spatial and temporal limits do *not* coincide.

Next, we make the following two important observations. Firstly, note that (25) yields that the support of the measure is

$$supp\,(\mu) = \left\{\frac{1}{1-\theta}, \frac{1}{1-\gamma}\right\}.$$

Therefore, the *spatial* limit coincides with the *right-end* of the support while the *temporal* limit with the *left-end* one.

Secondly, for each $x_0 > 0$ the temporal limit of the ratio $\frac{h^{(-1)}(x_0,t)}{t}$ is equal to *half* of the *left-end* point, since (28) yields

$$\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}$$

$$= \lim_{t\uparrow\infty}\left(\frac{1}{1-\theta} + \frac{1-\theta}{t}\ln\left(\frac{1}{2}\left(\sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t} + 4x} - \sqrt{e^{\left(\frac{1}{1-\theta}\right)^2 t}}\right)\right)\right) = \frac{1}{2\,(1-\theta)}.$$

In section 4 we will show that these two properties are always valid. In particular, we will see that it is the limit of the above ratio that plays the key role in establishing the temporal turnpike limit for general measures.

To juxtapose the above results with the ones in the traditional expected terminal utility setting, we compute the analogous quantities and associated limits for the cases analyzed in [10] and [2] for log-normal markets. Without loss of generality, we consider a market with a riskless bond of zero interest rate and a single log-normal stock with mean rate of return μ and volatility σ.

To this end, we fix an arbitrary horizon $T>0$ and, in analogy to (26), we take the *terminal* inverse marginal utility, $I(x)=(U')^{(-1)}(x)$, to be

$$I(x)=x^{-\frac{1}{1-\theta}}+x^{-\frac{1}{1-\gamma}},$$

for $x>0$ and θ,γ as in (25). This corresponds to terminal marginal utility $U'(x)=\left(\frac{\sqrt{1+4x}-1}{2}\right)^{\theta-1}$ and, thus, in analogy to (27),

$$\lim_{x\uparrow\infty}\frac{U'(x)}{x^{\gamma-1}}=1.$$

We now consider the value function, say $u(x,t;T)$ of the associated Merton problem, for $t\in[0,T]$. Letting $\tau=T-t$ be the time to the end of the investment horizon, we deduce, using well known results, that the function $\tilde{u}(x,\tau)\equiv u(x,T-t;T)$, satisfies, for $(x,\tau)\in\mathbb{R}_+\times[0,T)$, the Hamilton-Jacobi-Bellman equation

$$\tilde{u}_\tau+\frac{1}{2}\lambda^2\frac{\tilde{u}_x^2}{\tilde{u}_{xx}}=0.$$

The inverse spatial marginal value function $\tilde{v}:\mathbb{R}_+\times[0,T)\to\mathbb{R}_+$ then solves

$$\tilde{v}_\tau=\frac{1}{2}\lambda^2x^2\tilde{v}_{xx}+\lambda^2x\tilde{v}_x,$$

with $\tilde{v}(x,0)=I(x)$. We easily deduce that

$$\tilde{v}(x,\tau)=e^{\alpha\tau}x^{-\alpha}+e^{\beta\tau}x^{-2\alpha},$$

with $\alpha=\frac{1}{2}\lambda^2\frac{\theta}{(1-\theta)^2}$ and $\beta=\lambda^2\frac{1+\theta}{(1-\theta)^2}$. Note that $\beta>2\alpha$.

Taking the spatial inverse of $\tilde{v}(x,\tau)$ yields

$$\tilde{u}_x(x,\tau)=\left(\frac{e^{\alpha\tau}+\sqrt{e^{2\alpha\tau}+4xe^{\beta\tau}}}{2x}\right)^{1-\theta}.$$

Therefore, the associated risk tolerance function is given by

$$\tilde{r}(x,\tau)=\frac{1}{1-\theta}\left(\frac{2x}{1+\sqrt{1+4xe^{(\beta-2\alpha)\tau}}}+\frac{8x^2}{\left(\sqrt{e^{(2\alpha-\beta)\tau}}+\sqrt{e^{(2\alpha-\beta)\tau}+4x}\right)^2}\right).$$

In turn, for each $\tau_0 > 0$ and $x_0 > 0$, we obtain, respectively, the spatial and the temporal limits,

$$\lim_{x\uparrow\infty} \frac{\tilde{r}(x,\tau_0)}{x} = \frac{1}{1-\theta} \qquad \text{and} \qquad \lim_{\tau\uparrow\infty} \frac{\tilde{r}(x_0,\tau)}{x_0} = \frac{1}{1-\theta}.$$

3 Spatial asymptotic results

We examine the spatial asymptotic behavior of the risk tolerance function as $x \uparrow \infty$, for each $t_0 \geq 0$, under asymptotic assumptions for large wealth levels of the investor's initial risk preferences. In accordance with similar assumptions in [10] and [2], we impose this asymptotic assumption on the marginal $u_0'(x)$ instead of the function itself.

Assumption 1: *The initial datum* u_0 satisfies, for some $\gamma < 1$,

$$\lim_{x\uparrow\infty} \frac{u_0'(x)}{x^{\gamma-1}} = 1. \tag{32}$$

The next result yields necessary and sufficient conditions on b, the right end of the support of the measure, for the above assumption to hold.

Lemma 2 *Assumption (32) holds if and only if the associated measure μ satisfies*

$$b = \frac{1}{1-\gamma} \qquad \text{and} \quad \mu\left(\left\{\frac{1}{1-\gamma}\right\}\right) = 1. \tag{33}$$

Proof. From (32), (17) and the fact that $h(x,0)$ is strictly increasing and of full range, we have

$$1 = \lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma-1}} = \lim_{z\uparrow\infty} \frac{u_x(h(z,0),0)}{(h(z,0))^{\gamma-1}} = \lim_{z\uparrow\infty} \left(\frac{h(z,0)}{e^{\frac{1}{1-\gamma}z}}\right)^{1-\gamma}. \tag{34}$$

Therefore, representation (19) gives

$$\lim_{z\uparrow\infty} \int_a^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) = 1. \tag{35}$$

If $a = b$, then (33) follows directly. If $a < b$, then, it must be that $a \leq \frac{1}{1-\gamma}$, otherwise, we get a contradiction. In turn, for $\varepsilon > 0$,

$$\int_a^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) \geq \int_{\frac{1}{1-\gamma}+\varepsilon}^b e^{z(y-\frac{1}{1-\gamma})}\mu(dy) \geq e^{\varepsilon z}\mu\left([\frac{1}{1-\gamma}+\varepsilon, b]\right). \tag{36}$$

Sending $\varepsilon \downarrow 0$ and using (35) yield that $\mu((\frac{1}{1-\gamma}, b]) = 0$, and thus, $\text{supp}(\mu) \subseteq (a, \frac{1}{1-\gamma}]$. Moreover, we have from (35) that

$$1 = \lim_{z\uparrow\infty} \int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{z(y-\frac{1}{1-\gamma})}\mu(dy) + \mu(\{\frac{1}{1-\gamma}\}) = \mu(\{\frac{1}{1-\gamma}\}),$$

and we conclude. The rest of the proof follows easily. ■

We next state the main spatial asymptotic result.

Proposition 3 *Suppose that the initial datum u_0 satisfies the asymptotic property (32). Then, for each $t_0 \geq 0$, the relative risk tolerance converges to the right-end of the support of the measure μ,*

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \frac{1}{1-\gamma}. \tag{37}$$

Proof. Let $t_0 \geq 0$. From representation (36) we have that

$$h\left(z,t_0\right) = \int_a^{\left(\frac{1}{1-\gamma}\right)^-} e^{zy-\frac{1}{2}t_0 y^2}\mu(dy) + e^{\frac{1}{1-\gamma}z-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0},$$

and, in turn, the dominated convergence theorem implies

$$\lim_{z\uparrow\infty} \frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0}} = 1. \tag{38}$$

Therefore, from (17), together with the strict monotonicity and full range of $h(z,t_0)$, we deduce that

$$\lim_{x\uparrow\infty} \frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} = 1, \tag{39}$$

since

$$\begin{aligned}\lim_{x\uparrow\infty} \frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} &= \lim_{z\uparrow\infty} \frac{e^{-z+\frac{t_0}{2}}}{h^{\gamma-1}(z,t_0)e^{-\frac{\gamma}{2(1-\gamma)}t_0}} \\ &= \lim_{z\uparrow\infty} \left(\frac{h(z,t_0)}{e^{\frac{1}{1-\gamma}z-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0}}\right)^{1-\gamma} = 1.\end{aligned}$$

Next, we claim that

$$\lim_{x\uparrow\infty} \frac{u_{xx}(x,t_0)}{x^{\gamma-2}e^{-\frac{\gamma}{2(1-\gamma)}t_0}} = \frac{1}{\gamma-1}. \tag{40}$$

To prove this, it suffices to show that for any $t_0 \geq 0$, $u_x(x,t_0)$ is convex since the above would then follow from the arguments in Lemma 3.1 (ii) in [10]. To this end, differentiating (17) yields

$$u_{xxx}\left(h(z,t_0),t_0\right)\left(h_z(z,t_0)\right)^2 + u_{xx}(h(z,t_0),t_0)h_{zz}(z,t_0) = e^{-z+\frac{t_0}{2}}. \tag{41}$$

The strict convexity of h and the strict concavity of u then give

$$u_{xxx}\left(h(z,t_0),t_0\right) > 0, \tag{42}$$

and using the strict monotonicity and full range of h we conclude.

Combining (39) and (40) yields

$$\lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = \lim_{x\uparrow\infty} \left(-\frac{u_x(x,t_0)}{xu_{xx}(x,t_0)}\right)$$

$$= \lim_{x\uparrow\infty}\left(-\frac{u_x(x,t_0)}{x^{\gamma-1}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}\left(\frac{u_{xx}(x,t_0)}{x^{\gamma-2}e^{-\frac{\gamma}{2(1-\gamma)}t_0}}\right)^{-1}\right)=\frac{1}{1-\gamma}.$$

■

We stress that assumption (32), or equivalently (33), *cannot* be weakened. Indeed, as we will see in example 6.2, where we take the measure to be the Lebesgue on $[a,b]$, and thus there is no mass at b, the spatial turnpike property does not hold.

Corollary 4 *Suppose that the initial datum u_0 satisfies the asymptotic property (32). Then, for each $t_0 \geq 0$,*

$$\lim_{x\uparrow\infty} r_x\left(x,t_0\right)=\frac{1}{1-\gamma}. \tag{43}$$

Proof. From (24) we have that, for each $t_0 \geq 0$, $\lim_{x\uparrow\infty} r_x\left(x,t_0\right)$ exists, and we easily conclude. ■

4 Temporal (turnpike) asymptotic results

We investigate the temporal asymptotic behavior of the relative risk tolerance as $t\uparrow\infty$, for each $x_0>0$, under asymptotic assumption of the initial marginal utility for large wealth levels. This is the genuine "turnpike" analogue of similar results in classical expected utility models and the main finding herein. It shows that the relative risk tolerance will converge to the *left-end* of the support of the underlying measure μ.

As in the spatial case, we first relate the properties of the measure to the asymptotic behavior of the initial (marginal) datum.

Assumption 2: *There exists $\gamma<1$, $\gamma\neq 0$, such that for all $\gamma'\in(\gamma,1)$,*

$$\lim_{x\uparrow\infty}\frac{u_0'\left(x\right)}{x^{\gamma'-1}}=0, \tag{44}$$

while, for all $\gamma''<\gamma$,

$$\lim_{x\uparrow\infty}\frac{u_0'\left(x\right)}{x^{\gamma''-1}}=\infty. \tag{45}$$

As we show next, the above assumption is directly related to a condition introduced in [11] and [5], for a discrete and a continuous-time case, respectively.

Lemma 5 *Assumption 2 is equivalent to the function $u_0'\left(x\right)$ varying regularly at infinity with exponent $\gamma-1$, i.e. for all $k>0$,*

$$\lim_{x\uparrow\infty}\frac{u_0'(kx)}{u_0'(x)}=k^{\gamma-1}. \tag{46}$$

Proof. We first show that condition (46) implies (44) and (45). We argue by contradiction. Suppose that (44) does not hold. Then, there exists $\gamma' \in (\gamma, 1)$ and $\varepsilon > 0$ such that for x large enough, $\frac{u_0'(x)}{x^{\gamma'-1}} > \varepsilon$. On the other hand, condition (46) implies that, for all $k > 0$ and x large enough, $\left|\frac{u_0'(kx)}{u_0'(x)k^{\gamma-1}} - 1\right| < \varepsilon$. Thus, for large enough x,

$$0 < \frac{u_0'(kx)}{(kx)^{\gamma\prime-1}} = \frac{u_0'(kx)}{u_0'(x)k^{\gamma-1}}\frac{u_0'(x)}{x^{\gamma'-1}}k^{\gamma-\gamma'} < (1+\varepsilon)\frac{u_0'(x)}{x^{\gamma'-1}}k^{\gamma-\gamma'}.$$

Since $\gamma - \gamma' < 0$, $\lim_{k\uparrow\infty} \frac{u_0'(kx)}{(kx)^{\gamma'-1}} = 0$, and we conclude. Working similarly, we establish (45).

Next, we show the reverse direction. Assume that (45) and (44) hold. Then, for all $\delta, k > 0$ and x large enough,

$$\frac{u_0'(kx)}{(kx)^{\gamma+\delta-1}} < 1 \qquad \text{and} \qquad \frac{x^{\gamma-\delta-1}}{u_0'(x)} < 1.$$

Multiplying these two equations and rearranging gives, for all $\delta > 0$,

$$\frac{u_0'(kx)}{u_0'(x)} < \frac{(kx)^{\gamma+\delta-1}}{x^{\gamma-\delta-1}} = k^{\gamma+\delta-1}x^{2\delta}.$$

Similarly, it follows from interchanging kx and x in the above two inequalities that

$$\frac{u_0'(kx)}{u_0'(x)} > \frac{(kx)^{\gamma-\delta-1}}{x^{\gamma+\delta-1}} = k^{\gamma-\delta-1}x^{-2\delta},$$

and condition (46) follows by sending first $\delta \downarrow 0$ and then $x \uparrow \infty$. ■

Assumption 2 is weaker than Assumption 1, and implies, as we show next, that the measure μ has support with right-end point at $\frac{1}{1-\gamma}$, but without necessarily having a mass therein.

Lemma 6 *Assumption 2 holds if and only if the measure μ has finite support with its right boundary at $\frac{1}{1-\gamma}$, namely,*

$$\inf\{y > 0 : \mu((y, \infty)) = 0\} = \frac{1}{1-\gamma}. \tag{47}$$

Proof. We show that Assumption 2 implies property (47). For each $\gamma' \in (\gamma, 1)$, we deduce from (44) that

$$0 = \lim_{x\uparrow\infty} \frac{u_x(x,0)}{x^{\gamma'-1}} = \lim_{z\uparrow\infty} \frac{u_x(h(z,0),0)}{(h(z,0))^{\gamma'-1}} = \lim_{z\uparrow\infty} \left(\frac{h(z,0)}{e^{\frac{z}{1-\gamma'}}}\right)^{1-\gamma'},$$

and, thus,

$$\lim_{z\uparrow\infty} \int_a^b e^{z\left(y-\frac{1}{1-\gamma'}\right)}\mu(dy) = 0. \tag{48}$$

Next, observe that if $b \geq 1$, then it will contradict the above limit, and thus we need to have $b < 1$. Assume now that there exists $\gamma' \in (\gamma, 1)$ with $b = \frac{1}{1-\gamma'}$. Then, for each $\tilde{\gamma} \in (\gamma, \gamma')$ we have $\frac{1}{1-\tilde{\gamma}} < \frac{1}{1-\gamma'}$ and the above gives, for ε small enough,

$$\lim_{z \uparrow \infty} \left(\int_a^{\left(\frac{1}{1-\tilde{\gamma}}+\varepsilon\right)^-} e^{z\left(y-\frac{1}{1-\tilde{\gamma}}\right)} \mu\left(dy\right) + \int_{\frac{1}{1-\tilde{\gamma}}+\varepsilon}^{b} e^{z\left(y-\frac{1}{1-\tilde{\gamma}}\right)} \mu\left(dy\right) \right) = 0.$$

Therefore, it must be that $\mu\left([\frac{1}{1-\tilde{\gamma}}+\varepsilon, b]\right) = 0$. Sending $\varepsilon \downarrow 0$, gives $\mu\left(\left(\frac{1}{1-\tilde{\gamma}}, b\right]\right) = 0$, which is a contradiction. Thus, we must have $b \leq \frac{1}{1-\gamma}$. Similarly, using (45) we obtain that $b \geq \frac{1}{1-\gamma}$, and, thus, $b = \frac{1}{1-\gamma}$.

To show the reverse direction, we first observe that property (47) and the dominated convergence theorem yield that, for any $\varepsilon > 0$,

$$\lim_{z \uparrow \infty} h(z, 0) e^{-\left(\frac{1}{1-\gamma}+\varepsilon\right)z} = \lim_{z \uparrow \infty} \int_a^{\frac{1}{1-\gamma}} e^{z\left(y-\left(\frac{1}{1-\gamma}+\varepsilon\right)\right)} \mu(dy) = 0.$$

Then, setting γ' such that $\frac{1}{1-\gamma'} = \frac{1}{1-\gamma} + \varepsilon$, we deduce (44) for all $\gamma' \in (\gamma, 1)$.

The rest of the proof follows easily and it is thus omitted. ■

We have so far established that under Assumption 2 the associated measure μ has a finite right boundary (but not necessarily a mass) at $\frac{1}{1-\gamma}$, and vice-versa.

We now turn our attention to the left boundary of the support, denoted by a, where

$$a := \inf\{y \geq 0 : \mu\left((0, y]\right) > 0\}. \tag{49}$$

In the upcoming proofs we will frequently use the identity

$$x_0 = \int_a^{\frac{1}{1-\gamma}} e^{y h^{(-1)}(x_0, t) - \frac{1}{2} y^2 t} \mu(dy), \tag{50}$$

for $x_0 > 0$, which follows directly from (19) for $b = \frac{1}{1-\gamma}$.

Lemma 7 *Let $h^{(-1)} : \mathbb{D}_+ \to \mathbb{R}$ be the spatial inverse of h, and a as in (49). Then, for each $x_0 > 0$, $\lim_{t \uparrow \infty} h_t^{(-1)}(x_0, t)$ exists and, moreover, for $t \geq 0$,*

$$\frac{a}{2} \leq h_t^{(-1)}(x_0, t) \leq \frac{1}{2\left(1-\gamma\right)}. \tag{51}$$

Proof. Let $x_0 > 0$ and observe that (18) yields

$$h_t^{(-1)}\left(x_0, t\right) = \frac{1}{2} \frac{h_{xx}\left(h^{(-1)}\left(x_0, t\right)\right)}{h_x\left(h^{(-1)}\left(x_0, t\right), t\right)} = \frac{1}{2} \frac{\int_a^{\frac{1}{1-\gamma}} y^2 e^{y h^{(-1)}(x_0, t) - \frac{1}{2} y^2 t} \mu(dy)}{\int_a^{\frac{1}{1-\gamma}} y e^{y h^{(-1)}(x_0, t) - \frac{1}{2} y^2 t} \mu(dy)}$$

and thus inequality (51) holds, for all $t \geq 0$.

To show that $\lim_{t\uparrow\infty} h_t^{(-1)}(x_0,t)$ exists, it suffices to show that $h_t^{(-1)}(x_0,t)$ is decreasing in time. Indeed, direct calculations yield

$$h_{tt}^{(-1)}(x_0,t) = -\frac{\int_a^{\frac{1}{1-\gamma}} \left(yh_t^{(-1)}(x_0,t) - \frac{1}{2}y^2\right)^2 e^{yh^{(-1)}(x_0,t)-\frac{1}{2}y^2 t}\mu(dy)}{\int_a^{\frac{1}{1-\gamma}} ye^{yh^{(-1)}(x_0,t)-\frac{1}{2}y^2 t}\mu(dy)} < 0. \quad (52)$$

Alternatively, differentiating $h\left(h^{(-1)}(x_0,t),t\right) = x_0$ twice yields, setting $z = h^{(-1)}(x_0,t)$,

$$h_{tt}^{(-1)}(x_0,t)h_x(z,t)+\left(h_t^{(-1)}(x_0,t)\right)^2 h_{xx}(z,t)+2h_t^{(-1)}(x_0,t)\,h_{xt}(z,t)+h_{tt}(z,t) = 0.$$

We have that both h_x, $h_{xx} > 0$, as it follows directly from (19) and differentiation. Furthermore, the above quadratic in $h_t^{(-1)}(x,t)$ remains positive, which would then yield that $h_{tt}^{(-1)}(x_0,t) < 0$. Indeed,

$$h_{xt}^2(z,t) - h_{xx}(z,t)\,h_{tt}(z,t) = h_{xxx}^2(z,t) - h_{xx}(z,t)\,h_{xxxx}(z,t) < 0,$$

as it follows from (20). ∎

We are now ready to present one of the main findings herein, which yields the limit as $t \uparrow \infty$ of the ratio $\frac{1}{t}h^{(-1)}(x_0,t)$. We show that it converges to half of the lower-end of the measure's support. Some related weaker results can be found in [21].

Proposition 8 *Let $h^{(-1)} : \mathbb{D}_+ \to \mathbb{R}$ be the spatial inverse of the function h (cf. (19)) and let a, b be the left and right end of the support, respectively, with $a = 0^+$ or $a > 0$, and $b < \infty$. Then, for each $x_0 > 0$, the following assertions hold.*

i) It holds that

$$\lim_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{t} = \frac{a}{2}. \quad (53)$$

ii) Let

$$\Delta(x_0,t) := \frac{h^{(-1)}(x_0,t)}{t} - \frac{a}{2}. \quad (54)$$

If $a > 0$, then

$$|\Delta(x_0,t)| \leq \frac{1}{at}\ln\left(\frac{\mu\left([a,\frac{1}{1-\gamma}]\right)}{x_0}\right), \quad \textit{if} \;\; \Delta(x_0,t) < 0, \quad (55)$$

and

$$x_0 \geq \mu([a, a+\Delta(x_0,t)])\, e^{\frac{1}{2}ta\Delta(x_0,t)}, \quad \textit{if} \;\; \Delta(x_0,t) > 0. \quad (56)$$

If $a = 0^+$, then $\Delta(x_0,t) > 0$, and, moreover, for each $\theta \in (0,1)$,

$$x_0 \geq \mu([\Delta(x_0,t), (1+\theta)\Delta(x_0,t)])\, e^{\frac{1}{2}t\left(1-\theta^2\right)\Delta^2(x_0,t)}. \quad (57)$$

Proof. *i).* Let $x_0>0$ fixed. Recall that $h_t^{(-1)}(x_0,t)>0$ (cf. (51)) and, thus, $\lim_{t\uparrow\infty}h^{(-1)}(x_0,t)$ exists. Moreover, rewriting (50) as

$$x_0=\int_a^{\frac{1}{1-\gamma}}e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{1}{2}y\right)}\mu(dy), \tag{58}$$

we see that $\lim_{t\uparrow\infty}h^{(-1)}(x_0,t)=\infty$, otherwise, sending $t\uparrow\infty$ we get a contradiction. In turn, from Lemma 7 and L' Hospital's rule, we deduce that

$$A(x_0):=\lim_{t\uparrow\infty}\frac{h^{(-1)}(x_0,t)}{t}=\lim_{t\uparrow\infty}h_t^{(-1)}(x_0,t), \tag{59}$$

and thus

$$\frac{a}{2}\le A(x_0)\le\frac{1}{2(1-\gamma)}. \tag{60}$$

Next, we claim that $A\left(x_0\right)<\frac{1}{2(1-\gamma)}$.

Let $a>0$. If $a=\frac{1}{1-\gamma}$, then $a=b$ and $h^{(-1)}\left(x_0,t\right)=\ln x_0^{1-\gamma}+\frac{1}{2}\frac{1}{1-\gamma}t$, and the result follows directly.

Let $0<a<\frac{1}{1-\gamma}$. Assume that there exists x_0 such that $A\left(x_0\right)=\frac{1}{2(1-\gamma)}$. Then, for $\varepsilon>0$, there exists $t_0\left(x_0,\varepsilon\right)$ such that, for $t\ge t_0$,

$$-\varepsilon\le\frac{h^{(-1)}(x_0,t)}{t}-\frac{1}{2(1-\gamma)}\le\varepsilon.$$

In turn, for $\delta>0$ small enough, the above and (50) yield

$$x_0\ge\int_a^{\left(\frac{1}{1-\gamma}-2\varepsilon-\delta\right)^-}e^{ty\left(\frac{1}{2(1-\gamma)}-\varepsilon-\frac{1}{2}y\right)}\mu(dy)+\int_{\frac{1}{1-\gamma}-2\varepsilon-\delta}^{\frac{1}{1-\gamma}}e^{ty\left(\frac{1}{2(1-\gamma)}-\varepsilon-\frac{1}{2}y\right)}\mu(dy),$$

which yields a contradiction as $t\uparrow\infty$, because the first integral would converge to ∞.

Next, assume that there exists $x_0>0$ such that

$$\frac{a}{2}<A(x_0)<\frac{1}{2(1-\gamma)}. \tag{61}$$

Then, for $\varepsilon,\delta>0$ small enough we have

$$a<2(A(x_0)-\varepsilon)-\delta<2(A(x_0)-\varepsilon)<\frac{1}{1-\gamma}. \tag{62}$$

From (50), we then deduce that, for $t\ge t_0(x_0,\varepsilon)$, $x_0\ge\int_a^{\frac{1}{1-\gamma}}e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy)$. If $\mu\left(\{a\}\right)\ne0$, then $x_0\ge e^{\frac{ta}{2}(2(A(x_0)-\varepsilon)-a)}\mu\left(\{a\}\right)$, and sending $t\uparrow\infty$ yields a contradiction. If $\mu\left(\{a\}\right)=0$, then

$$x_0\ge\int_a^{\frac{1}{1-\gamma}}e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy)\ge\int_a^{2(A(x_0)-\varepsilon)-\delta}e^{t(y(A(x_0)-\varepsilon)-\frac{1}{2}y^2)}\mu(dy). \tag{63}$$

Consider the quadratic $B\left(y\right) = y(A(x_0) - \varepsilon) - \frac{1}{2}y^2$. We have

$$B\left(y_1\right) = B\left(y_2\right) = 0, \text{ for } \ y_1 = 0 \ \text{ and } \ y_2 = 2\left(A(x_0) - \varepsilon\right),$$

$B\left(y\right) > 0$, for $0 < y < 2\left(A(x_0) - \varepsilon\right)$, and $B\left(y\right)$ achieves a maximum at $y^* = A(x_0) - \varepsilon$.

Next, we look at its minimum, $y_* = \min_{a \leq y \leq 2(A(x) - \varepsilon) - \delta} \Delta\left(y\right)$, and claim that

$$y_* = 2(A(x_0) - \varepsilon) - \delta. \tag{64}$$

Indeed, if $0 < a \leq y^*$, then choosing $\delta < a$, direct calculations yield $\Delta\left(a\right) > \Delta\left(y_*\right)$. If $y^* < a$, then (62) yields $a < y_* < y_2$, and, thus, the minimum also occurs at y_*.

Clearly, because $y_1 < y_* < y_2$, we have $B\left(y_*\right) = \frac{1}{2}\delta\left(2(A(x_0) - \varepsilon) - \delta\right) > 0$. Therefore, for $t \geq t_0(x_0, \varepsilon)$,

$$x_0 \geq \int_a^{2(A(x_0) - \varepsilon) - \delta} e^{tB(y_*)} \mu(dy). \tag{65}$$

As $t \uparrow \infty$, the right hand side of (65) converges to ∞, unless it holds that $\mu\left([a, 2(A(x_0) - \varepsilon) - \delta]\right) = 0$. Sending $\delta \downarrow 0$ and $\varepsilon \downarrow 0$, we then have

$$\mu([a, 2A(x_0)]) = 0,$$

which, however, contradicts (61). Therefore, it must be that that, for all $x > 0$, $A(x_0) \leq \frac{a}{2}$, and we easily conclude.

If $a = 0^+$, similar arguments yield that for every $\theta \in \left(0, A\left(x_0\right)\right]$, we have that $\mu([\theta, 2A(x_0)]) = 0$. Sending $\theta \downarrow 0$ yields $\mu\left(0, 2A\left(x_0\right)\right] = 0$, which contradicts (61).

ii). Let $a > 0$.

If $\Delta\left(x_0, t\right) < 0$, from (50) we have

$$x_0 = \int_a^{\frac{1}{1-\gamma}} e^{ty\left(\Delta(x_0,t) + \frac{1}{2}(a-y)\right)} \mu(dy)$$

$$\leq e^{ta\Delta(x_0,t)} \int_a^{\frac{1}{1-\gamma}} e^{\frac{1}{2}ty(a-y)} \mu\left(dy\right) \leq e^{ta\Delta(x_0,t)} \mu\left(\left[a, \frac{1}{1-\gamma}\right]\right),$$

and (55) follows.

If $\Delta\left(x_0, t\right) > 0$, then (53) yields that, for ε small enough and $t \geq t_0\left(x_0, \varepsilon\right)$, $0 < \frac{h^{(-1)}(x_0,t)}{t} - \frac{a}{2} < \varepsilon$. Choosing ε such that $\varepsilon < \frac{1}{2(1-\gamma)} - \frac{a}{2}$ yields $0 < \frac{h^{(-1)}(x_0,t)}{t} - \frac{a}{2} < \frac{1}{2(1-\gamma)} - \frac{a}{2}$, and using that $a < \frac{1}{1-\gamma}$, gives

$$\frac{a}{2} + \frac{h^{(-1)}\left(x_0, t\right)}{t} \leq \frac{1}{1-\gamma}.$$

From (28) we then deduce that

$$x_0 \geq \int_a^{\frac{a}{2}+\frac{h(-1)(x_0,t)}{t}} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{y}{2}\right)}\mu\left(dy\right).$$

The quadratic $H\left(y\right) := y\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{y}{2}\right)$ in the above integrand becomes zero at $y_1 = 0$ and $y_3 = 2\frac{h^{(-1)}(x_0,t)}{t} > a$ and, therefore, its minimum occurs at one of the end points a or $\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}$. Note that $a < \frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t} < y_3$.

If it occurs at a, then $H\left(a\right) = a\Delta\left(x_0,t\right)$, while if it occurs at $\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}$, then $H\left(\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}\right) = \frac{1}{2}\left(\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}\right)\Delta\left(x_0,t\right) > \frac{1}{2}a\Delta\left(x_0,t\right).$

Combining the above gives

$$x_0 \geq \int_a^{\frac{a}{2}+\frac{h^{(-1)}(x_0,t)}{t}} e^{\frac{1}{2}ta\Delta(x_0,t)}\mu\left(dy\right) = \mu\left(\left[a, a+\Delta\left(x_0,t\right)\right]\right)e^{\frac{1}{2}ta\Delta(x_0,t)}.$$

Finally, let $a = 0^+$. Then, $\Delta\left(x_0,t\right) = \frac{h^{(-1)}(x_0,t)}{t}$.

Recall that $\lim_{t\uparrow\infty} h^{(-1)}\left(x_0,t\right) = \infty$, and thus $\frac{h^{(-1)}(x_0,t)}{t} > 0$, for t large. For $\varepsilon \in \left(\frac{h^{(-1)}(x_0,t)}{t}, 2\frac{h^{(-1)}(x_0,t)}{t}\right)$ we then have

$$x_0 \geq \int_{\frac{h^{(-1)}(x_0,t)}{t}}^{\varepsilon} e^{ty\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{y}{2}\right)}\mu\left(dy\right) \geq \int_{\frac{h^{(-1)}(x_0,t)}{t}}^{\varepsilon} e^{t\varepsilon\left(\frac{h^{(-1)}(x_0,t)}{t}-\frac{\varepsilon}{2}\right)}\mu\left(dy\right).$$

Setting $\varepsilon = \left(1+\theta\right)\frac{h^{(-1)}(x_0,t)}{t}$, (57) follows. ■

We are now ready to prove one of the main results herein.

Theorem 9 *Let a be the left end of the support of the measure μ. Then, for each $x_0 > 0$,*

$$\lim_{t\uparrow\infty}\frac{r\left(x_0,t\right)}{x_0} = a. \tag{66}$$

Furthermore, there exists a function $G\left(x_0,t\right)$ given by

$$G\left(x_0,t\right) := \begin{cases} \int_a^{\frac{1}{1-\gamma}}(y-a)e^{-ty(\frac{y-a}{2})}\mu(dy), \quad \Delta\left(x_0,t\right) < 0 \\ 2\Delta\left(x_0,t\right)x_0 + \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}}(y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy), \ \Delta\left(x_0,t\right) > 0, \end{cases}$$

satisfying with $\lim_{t\uparrow\infty} G\left(x_0,t\right) = 0$ and, for t large enough,

$$0 \leq r(x_0,t) - ax_0 \leq G\left(x_0,t\right). \tag{67}$$

Proof. We present two alternative convergence proofs. The first yields (66) while the second gives the rate of convergence $G(x_0, t)$.

To this end, differentiating (17) gives

$$u_{xt}(x_0, t) = \left(\frac{1}{2} - h_t^{(-1)}(x_0, t)\right) u_x(x_0, t). \tag{68}$$

Moreover, (14) and (16) imply that $u_t(x_0, t) = -\frac{1}{2}u_x(x_0, t)r(x_0, t)$ and, in turn,

$$u_{tx}(x_0, t) = -\frac{1}{2}u_{xx}(x_0, t)\, r(x_0, t) - \frac{1}{2}u_x(x_0, t)\, r_x(x_0, t). \tag{69}$$

Combining the above we deduce

$$\frac{1}{2}r_x(x_0, t) = h_t^{(-1)}(x_0, t), \tag{70}$$

and from Proposition 8 and (59)

$$\lim_{t\uparrow\infty} r_x(x_0, t) = \lim_{t\uparrow\infty} 2h_t^{(-1)}(x_0, t) = a. \tag{71}$$

On the other hand,

$$\lim_{c\downarrow 0^+} \int_c^{x_0} r_x(\rho, t)d\rho = r(x_0, t) - \lim_{c\downarrow 0^+} r(c, t).$$

Using the fact that, for all $t \geq 0$, $\lim_{x\downarrow 0^+} r(x, t) = 0$ (see [18]), we get that, for $x_0 > 0$,

$$r(x_0, t) = \int_a^{x_0} r_x(\rho, t)d\rho. \tag{72}$$

Finally, we deduce from (70) and (52) that $r_{xt}(x_0, t) < 0$, and thus, for $x_0 > 0$, we have for $y \in (0, x_0]$, $r_x(y, t) \leq r_x(x_0, 0)$. However, for all $x_0 > 0$, $r_x(x_0, 0) < \infty$. This follows directly from (21),(19) and the full range of $h(x, 0)$, since

$$r_x(h(z, 0), 0) = \frac{h_{zz}(z, 0)}{h_z(z, 0)} = \frac{\int_a^{\frac{1}{1-\gamma}} y^2 e^{yz - \frac{1}{2}t^2 y}\mu(dy)}{\int_a^{\frac{1}{1-\gamma}} y e^{yz - \frac{1}{2}t^2 y}\mu(dy)} \leq \frac{1}{1-\gamma}.$$

Using the dominated convergence theorem and passing to the limit as $t \uparrow \infty$ in (70), we deduce (66).

Next, we give the second convergence proof, which also yields the rate of convergence. First note that

$$0 \leq r(x_0, t) - ax_0. \tag{73}$$

This follows directly from (21), (19) and (50), for

$$r(x_0, t) = \int_a^{\frac{1}{1-\gamma}} y e^{t(y\frac{h^{(-1)}(x_0, t)}{t} - \frac{1}{2}y^2)}\mu(dy) \geq a \int_a^{\frac{1}{1-\gamma}} e^{t(y\frac{h^{(-1)}(x_0, t)}{t} - \frac{1}{2}y^2)}\mu(dy).$$

Furthermore, from (21), (19), (50) and (54), we have

$$r(x_0,t) - ax_0 = \int_a^{\frac{1}{1-\gamma}} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy). \tag{74}$$

If $\Delta(x_0,t) < 0$ (which occurs only if $a > 0$, as shown in the previous proof), then the above yields

$$r(x_0,t) - ax_0 \le \int_a^{\frac{1}{1-\gamma}} (y-a)e^{-ty(\frac{y-a}{2})}\mu(dy),$$

and (67) follows directly with $G(t) := \int_a^{\frac{1}{1-\gamma}} (y-a)e^{-ty(\frac{y-a}{2})}\mu(dy)$.

Let $\Delta(x_0,t) > 0$ and $a > 0$ or $a = 0^+$. If $a = \frac{1}{1-\gamma}$, then the result follows trivially.

For $a < \frac{1}{1-\gamma}$, observe that for t large enough, $0 < a + 2\Delta(x_0,t) < \frac{1}{1-\gamma}$, and thus representation (74) gives

$$r(x_0,t) - ax_0 = \int_a^{(a+2\Delta(x_0,t))^-} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy)$$

$$+ \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy).$$

Let $C_1(x_0,t) := \int_a^{(a+2\Delta(x_0,t))^-} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy)$, and observe that

$$C_1(x_0,t) \le 2\Delta(x_0,t) \int_a^{(a+2\Delta(x_0,t))^-} e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy) \le 2\Delta(x_0,t)\, x_0,$$

where we used (50). Thus

$$\lim_{t\uparrow\infty} C_1(x_0,t) = 0. \tag{75}$$

Let also $C_2(x_0,t) := \int_{a+2\Delta(x_0,t)}^{\frac{1}{1-\gamma}} (y-a)e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}\mu(dy)$ and $F(y,t,x_0) := (y-a)\, e^{ty(\frac{2\Delta(x_0,t)+a-y}{2})}$, $y \in \left[a + 2\Delta(x_0,t), \frac{1}{1-\gamma}\right]$. Then, $F(a+2\Delta(x_0,t), t, x_0) = 2\Delta(x_0,t)$, and thus $\lim_{t\uparrow\infty} F(a+2\Delta(x_0,t), t, x_0) = 0$. Furthermore, for each $y \in \left(a + 2\Delta(x_0,t), \frac{1}{1-\gamma}\right]$, we also have $\lim_{t\uparrow\infty} F(y,t,x_0) = 0$. In turn, the dominated convergence theorem gives

$$\lim_{t\uparrow\infty} C_2(x_0,t) = 0. \tag{76}$$

Setting $G(x_0,t) := C_1(x_0,t) + C_2(x_0,t)$, and using (75) and (76), we obtain (67). ■

5 Spatial and temporal limits for the relative prudence function

We now revert our attention to the relative prudence function $p(x,t)$ defined, for $(x,t) \in \mathbb{D}_+$, as

$$p(x,t) = -\frac{x u_{xxx}(x,t)}{u_{xx}(x,t)}, \tag{77}$$

with u solving (14).

Proposition 10 *For $(x,t) \in \mathbb{D}_+$, we have that $p(x,t) > 0$. Moreover, the following spatial and temporal limits hold.*

i) If Assumption 1 holds, then, for each $t_0 \geq 0$,

$$\lim_{x\uparrow\infty} p(x,t_0) = 2 - \gamma. \tag{78}$$

ii) If Assumption 2 holds, then, for each $x_0 > 0$,

$$\lim_{t\uparrow\infty} p(x_0,t) = \begin{cases} 1 + \frac{1}{a}, & \text{if } \ a > 0 \\ \\ \infty, & \text{if } a = 0^+. \end{cases} \tag{79}$$

Proof. Using (77) and (16), we deduce that, for each $t_0 \geq 0$,

$$p(x,t_0) = \frac{x}{r(x,t_0)}\left(1 + r_x(x,t_0)\right),$$

and the fact that $p(x,t_0) > 0$ and (78) follow directly from (23) and (37), respectively.

From (77) and equation (14) we also obtain that, for each $x_0 > 0$,

$$\frac{u_{xt}(x_0,t)}{u_x(x_0,t)} = 1 - \frac{1}{2}\frac{r(x_0,t)}{x_0} p(x_0,t) = \frac{1}{2} - h_t^{(-1)}(x_0,t). \tag{80}$$

Using that $\lim_{t\uparrow\infty} h_t^{(-1)}(x_0,t) = \frac{a}{2}$ we easily conclude. ■

6 Examples

We present two representative examples in which the measure is, respectively, a sum of Dirac functions and the Lebesgue measure. The first example generalizes the results of the example in subsection 2.1, while the second demonstrates that the spatial turnpike property fails if there is no mass at the right end of the measure's support.

6.1 Finite sum of Dirac functions

We assume that

$$\mu = \sum_{n=1}^{N} \delta_{y_n}, \quad \text{with} \quad 0 < y_1 < \cdots < y_N = \frac{1}{1-\gamma}.$$

Then, $h(z,0) = \Sigma_{n=1}^{N} e^{y_n z}$ and, thus, $\lim_{z\uparrow\infty} h\left(z,0\right) e^{-zy_N} = 1$. In turn, (34) yields

$$\lim_{x\uparrow\infty} \frac{u_x\left(x,0\right)}{x^{\gamma-1}} = 1,$$

which verifies the results of Lemma 2. We also have, for $(z,t) \in \mathbb{D}$,

$$h(z,t) = \sum_{n=1}^{N} \exp\left(y_n z - \frac{1}{2} y_n^2 t\right).$$

(cf. (19)), and, therefore, for $x > 0$,

$$x = \sum_{n=1}^{N} \exp\left(y_n t \left(\frac{h^{(-1)}(x,t)}{t} - \frac{1}{2} y_n\right)\right). \tag{81}$$

Furthermore,

$$h^{(-1)}(x,t) - \frac{1}{2} y_1 t \leq \frac{1}{y_1} \ln x. \tag{82}$$

6.1.1 Temporal asymptotic expansion of $h^{(-1)}(x_0,t)$ for large t

We claim that, for each $x_0 > 0$, as $t \uparrow \infty$,

$$h^{(-1)}(x_0,t) = \frac{1}{2} y_1 t + \frac{1}{y_1} \ln x_0 + o(1). \tag{83}$$

Indeed, using the limit (53), we have

$$\lim_{t\uparrow\infty} \left(\frac{h^{(-1)}(x_0,t)}{t} - \frac{1}{2} y_n\right) \begin{cases} < 0, & 1 < n \leq N \\ = 0, & n = 1. \end{cases}$$

Therefore, as $t \uparrow \infty$, all the terms in (81) vanish except for the first one, and thus,

$$x_0 = \lim_{t\uparrow\infty} \exp\left(y_1 h^{(-1)}(x_0,t) - \frac{1}{2} y_1^2 t\right). \tag{84}$$

Taking logarithm and rearranging terms yields (83).

6.1.2 Spatial asymptotic expansion of $h^{(-1)}(x,t_0)$ for large x

We claim that, for each $t_0 \geq 0$,

$$h^{(-1)}(x,t_0) = (1-\gamma)\ln x + \frac{1}{2\,(1-\gamma)}t_0 + o(1). \tag{85}$$

To obtain this, we first establish that

$$\lim_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} = (1-\gamma)\,. \tag{86}$$

Indeed, fix $t_0 \geq 0$, let $\delta \in (0, \frac{1}{1-\gamma})$ and assume that

$$\liminf_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} < \frac{1}{\frac{1}{1-\gamma}+\delta}.$$

Then, using (81) and that $h^{(-1)}(x,t_0) > 0$, for large x, we have

$$1 = \frac{1}{x}\sum_{n=1}^{N} \exp\left(y_n \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x}\right) - \frac{1}{2}y_n^2 t_0\right)$$

$$\leq \frac{1}{x}\sum_{n=1}^{N} \exp\left(y_n \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x}\right)\right) \leq N x^{\frac{1}{1-\gamma}\frac{h^{(-1)}(x,t_0)}{\ln x}-1},$$

and using that $\frac{1}{1-\gamma}\frac{1}{\frac{1}{1-\gamma}+\delta} - 1 = -\frac{\delta(1-\gamma)}{1+\delta(1-\gamma)} < 0$, we get a contradiction as $x \uparrow \infty$.

Since δ is arbitrary, we deduce that

$$\liminf_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} \geq (1-\gamma)\,. \tag{87}$$

Similarly, assume that for $\delta \in \left(0, \frac{1}{1-\gamma}\right)$,

$$\limsup_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} > \frac{1}{\frac{1}{1-\gamma}-\delta}.$$

Then, (82) gives

$$1 > \frac{1}{x}\exp\left(\frac{1}{1-\gamma}\ln x \frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0\right)$$

$$= x^{\frac{1}{1-\gamma}\frac{h^{(-1)}(x,t_0)}{\ln x}-1} e^{-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2 t_0}$$

and using that $\frac{1}{1-\gamma}\frac{1}{\frac{1}{1-\gamma}-\delta} - 1 = \frac{\delta(1-\gamma)}{1-\delta(1-\gamma)} > 0$, we get a contradiction as $x \uparrow \infty$. Since δ is arbitrary, we deduce that

$$\limsup_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{\ln x} \leq (1-\gamma)\,, \tag{88}$$

and we easily conclude.

Next, we rewrite (81) as

$$1 = \sum_{n=1}^{N} \exp\left(y_n h^{(-1)}(x,t_0) - \frac{1}{2} y_n^2 t_0 - \ln x \right) \tag{89}$$

$$= \sum_{n=1}^{N} \exp\left(y_n \ln x \left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{y_n} \right) - \frac{1}{2} y_n^2 t_0 \right).$$

Note that from the limit in (86) we have that

$$\lim_{x\uparrow\infty} \left(\frac{h^{(-1)}(x,t_0)}{\ln x} - \frac{1}{y_n} \right) = \begin{cases} < 0, & 1 \le n < N \\ = 0, & n = N. \end{cases}$$

Therefore, as $x \uparrow \infty$, the first $N-1$ terms in (89) vanish, and we deduce that

$$\lim_{x\uparrow\infty} \exp\left(\frac{1}{1-\gamma} h^{(-1)}(x,t_0) - \ln x - \frac{1}{2} \left(\frac{1}{1-\gamma} \right)^2 t_0 \right) = 1.$$

We then obtain (85) by taking the logarithm and rearranging the terms.

6.1.3 Spatial and temporal asymptotics of $r(x,t)$

From representation (21), we have for the risk tolerance function

$$r(x,t) = \sum_{n=1}^{N} y_n \exp\left(y_n h^{(-1)}(x,t) - \frac{1}{2} y_n^2 t \right). \tag{90}$$

Let $x_0 > 0$. Then, (82) gives

$$r(x_0,t) \le \sum_{n=1}^{N} y_n \exp\left(y_n (\frac{1}{2} y_1 t + \frac{1}{y_1} \ln x_0) - \frac{1}{2} y_n^2 t \right)$$

$$= y_1 x_0 + \sum_{n=2}^{N} y_n \exp\left(\frac{1}{2} y_n (y_1 - y_n) t \right) x_0^{\frac{y_n}{y_1}}.$$

Therefore, the temporal asymptotic expansion of $r(x_0,t)$ as $t \uparrow \infty$ is given by

$$r(x_0,t) = y_1 x_0 + O\left(e^{\frac{1}{2} y_2 (y_1 - y_2) t} \right). \tag{91}$$

Next, let $t_0 \ge 0$. Then,

$$\lim_{x\uparrow\infty} r(x,t_0) = \lim_{x\uparrow\infty} \sum_{n=1}^{N} y_n \exp\left(y_n ((1-\gamma)\ln x + \frac{1}{2(1-\gamma)} t_0) - \frac{1}{2} y_n^2 t_0 \right),$$

and, thus, as $x \uparrow \infty$,

$$r(x,t_0) = \sum_{n=1}^{N} y_n \exp\left(\frac{1}{2} y_n t_0 (\frac{1}{1-\gamma} - y_n)\right) x^{(1-\gamma)y_n} + o(1). \tag{92}$$

Therefore, for each $x_0 > 0$ and $t_0 \geq 0$, we have the temporal asymptotic expansion (91) yields

$$\lim_{t\uparrow\infty} \frac{r(x_0,t)}{x_0} = y_1 \quad \text{and} \quad \lim_{x\uparrow\infty} \frac{r(x,t_0)}{x} = y_N = \frac{1}{1-\gamma},$$

and these limits are consistent with the findings in Proposition 3 and Theorem 9, respectively.

6.2 Lebesgue measure

We consider a case of a measure with continuous support but without a mass at its right boundary. We derive the associated limits and also show that the spatial turnpike property fails.

- Lebesgue measure on $\left[a, \frac{1}{1-\gamma}\right]$, $a > 0$

Consider the functions $\varphi(z) := e^{-\frac{z^2}{2}}$ and $\Phi(z) := \int_{-\infty}^{z} \varphi(y)dy$, for $z \in \mathbb{R}$. Then, representations (19) and (50) yield, respectively,

$$h(z,t) = \int_a^{\frac{1}{1-\gamma}} e^{yz - \frac{1}{2}y^2 t} dy = \frac{e^{z^2/2t}}{\sqrt{t}} \int_{a\sqrt{t}-z/\sqrt{t}}^{\frac{1}{1-\gamma}\sqrt{t}-z/\sqrt{t}} \varphi(y)dy, \tag{93}$$

and

$$x = \int_a^{\frac{1}{1-\gamma}} e^{yt\left(\frac{h^{(-1)}(x,t)}{t} - \frac{1}{2}y\right)} dy = \frac{1}{\sqrt{t}} e^{\frac{h^{(-1)}(x,t)^2}{2t}} \int_{a\sqrt{t}-\frac{h^{(-1)}(x,t)}{\sqrt{t}}}^{\frac{1}{1-\gamma}\sqrt{t}-\frac{h^{(-1)}(x,t)}{\sqrt{t}}} \varphi(y)dy. \tag{94}$$

6.2.1 Temporal asymptotic expansion of $h^{(-1)}(x_0,t)$ for large t

We claim that for $x_0 > 0$, as $t \uparrow \infty$,

$$h^{(-1)}(x_0,t) = \frac{1}{2}at + \frac{1}{a}\left(\ln t + \ln x_0 + \ln \frac{a}{2}\right) + o(1). \tag{95}$$

To show this, we first establish that

$$x_0 = \lim_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{\frac{1}{2}at}. \tag{96}$$

Using (94) and that, for $z < 0$,

$$\Phi(z) \leq -\frac{\varphi(z)}{z}, \tag{97}$$

we have, for t large enough,

$$
\begin{aligned}
x_0 &\leq \frac{1}{\sqrt{t}} \exp\left(\frac{h^{(-1)}(x_0,t)^2}{2t}\right) \Phi\left(-a\sqrt{t} + \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right) \\
&\leq \frac{1}{\sqrt{t}} \frac{1}{a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}} \exp\left(\frac{h^{(-1)}(x_0,t)^2}{2t}\right) \varphi\left(-a\sqrt{t} + \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right) \\
&= \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{at - h^{(-1)}(x_0,t)}.
\end{aligned}
$$

In turn,

$$
x_0 \leq \liminf_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{at - h^{(-1)}(x_0,t)}. \tag{98}
$$

Next, we show that

$$
x_0 \geq \limsup_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)}}{at - h^{(-1)}(x_0,t)},
$$

which with (98) will yield (96). To this end, we use that for any $b > a > 0$, the inequality

$$
\Phi(b) - \Phi(a) \geq \frac{1}{b}\left(\varphi(a) - \varphi(b)\right)
$$

holds. Let $1 < k < \frac{1}{a(1-\gamma)}$. From (94) and the above, we have, for t large enough, that

$$
\begin{aligned}
x_0 \geq \frac{1}{\sqrt{t}} e^{\frac{h^{(-1)}(x_0,t)^2}{2t}} &\left(\Phi\left(ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right) - \Phi\left(a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)\right) \\
&\geq \frac{1}{\sqrt{t}} \frac{1}{ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}} e^{\frac{h^{(-1)}(x_0,t)^2}{2t}} \\
&\times \left(\varphi\left(a\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right) - \varphi\left(ka\sqrt{t} - \frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)\right) \\
&= \frac{1}{kat - h^{(-1)}(x_0,t)} \left(e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)} - e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}\right).
\end{aligned}
$$

From Proposition 8 and since $k > 1$, we have

$$
\lim_{t\uparrow\infty} \frac{e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}}{kat - h^{(-1)}(x_0,t)} = \lim_{t\uparrow\infty} \frac{e^{ka^2 t(\frac{h^{(-1)}(x_0,t)}{at} - \frac{k}{2})}}{at\left(k - \frac{h^{(-1)}(x_0,t)}{at}\right)} = 0.
$$

Therefore,

$$
x_0 \geq \limsup_{t\uparrow\infty} \frac{1}{kat - h^{(-1)}(x_0,t)} \left(e^{a(h^{(-1)}(x_0,t) - \frac{1}{2}at)} - e^{ka(h^{(-1)}(x_0,t) - \frac{1}{2}kat)}\right)
$$

$$\geq \limsup_{t\uparrow\infty} \frac{e^{ka(h^{(-1)}(x_0,t)-\frac{1}{2}kat)}}{kat-h^{(-1)}(x_0,t)} - \lim_{t\uparrow\infty} \frac{e^{ka(h^{(-1)}(x_0,t)-\frac{1}{2}kat)}}{kat-h^{(-1)}(x_0,t)} = \limsup_{t\uparrow\infty} \frac{e^{a(h^{(-1)}(x_0,t)-\frac{1}{2}at)}}{kat-h^{(-1)}(x_0,t)},$$

and sending $k \downarrow 1$ we conclude.

Next, we utilize the Lambert-W function $W(x)$, defined as the inverse function of $F(x) = xe^x$, to derive the explicit asymptotic expansion of $h^{(-1)}(x_0, t)$ as $t \uparrow \infty$. Recalling the notation $\Delta(x_0, t) = h^{(-1)}(x_0, t) - \frac{1}{2}at$, we deduce from (96) that there exists $\varepsilon(t)$ with $\lim_{t\uparrow\infty} \varepsilon(t) = 0$, such that

$$\frac{e^{a\Delta(x_0,t)}}{\frac{1}{2}at - \Delta(x_0,t)} = x_0(1+\varepsilon(t)).$$

Rewriting it yields

$$a\left(\frac{1}{2}at - \Delta(x_0,t)\right) e^{a(\frac{1}{2}at-\Delta(x_0,t))} = \frac{a}{x_0(1+\varepsilon(t))} e^{\frac{1}{2}a^2 t},$$

Using that the left hand side is of the form $F(a(\frac{1}{2}at - \Delta(x_0,t)))$, we obtain

$$a(\frac{1}{2}at - \Delta(x_0,t)) = W\left(\frac{a}{x_0(1+\varepsilon(t))} e^{\frac{1}{2}a^2 t}\right),$$

and, in turn,

$$\Delta(x_0,t) = \frac{1}{2}at - \frac{1}{a} W\left(\frac{a}{x_0(1+\varepsilon(t))} e^{\frac{1}{2}a^2 t}\right).$$

It is established in [3] that the asymptotic expansion of $W(x)$, for large x, is given by

$$W(x) = \ln x - \ln(\ln x) + o(1).$$

Therefore,

$$\begin{aligned}
\Delta(x_0,t) &= \frac{1}{2}at - \frac{1}{a}\ln\left(\frac{a}{x_0(1+\varepsilon(t))} e^{\frac{1}{2}a^2 t}\right) + \frac{1}{a}\ln\ln\left(\frac{a}{x_0(1+\varepsilon(t))} e^{\frac{1}{2}a^2 t}\right) + o(1) \\
&= \frac{1}{a}\left(\ln\frac{x_0}{a} + \ln(1+\varepsilon(t)) + \ln\left(\frac{1}{2}a^2 t + \ln\frac{a}{x_0(1+\varepsilon(t))}\right)\right) + o(1).
\end{aligned}$$

Using that as $t \uparrow \infty$, $\ln(1+\varepsilon(t)) = o(1)$ and that

$$\ln\left(\frac{1}{2}a^2 t + \ln\frac{a}{x_0(1+\varepsilon(t))}\right) = \ln\left(\frac{1}{2}a^2 t\right) + o(1),$$

assertion (95) follows.

6.2.2 Spatial asymptotic expansion of $h^{(-1)}(x, t_0)$ for large x

Let $t_0 \geq 0$. We show that, as $x \uparrow \infty$,

$$h^{(-1)}(x,t_0) = \frac{1}{2(1-\gamma)} t_0 + (1-\gamma)\left(\ln x + \ln\ln x - \ln\frac{1}{1-\gamma}\right) + o(1). \quad (99)$$

We first establish that

$$\lim_{x\uparrow\infty}\frac{h^{(-1)}(x,t_0)}{\ln x}=(1-\gamma)\,. \tag{100}$$

Indeed, let $f(z,t):=\frac{1}{z}e^{\frac{1}{1-\gamma}z-\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2t}$. Then,

$$\lim_{z\uparrow\infty}\frac{h(z,t_0)}{f(z,t_0)}=\lim_{z\uparrow\infty}\int_a^{\frac{1}{1-\gamma}}ze^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy$$

$$=\lim_{z\uparrow\infty}\left(\int_a^{\frac{1}{1-\gamma}}(z-yt_0)e^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy+\int_a^{\frac{1}{1-\gamma}}yt_0e^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right)$$

$$=\lim_{z\uparrow\infty}\left(1-e^{(a-\frac{1}{1-\gamma})z-\frac{1}{2}(a^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}+\int_a^{\frac{1}{1-\gamma}}yt_0e^{z(y-\frac{1}{1-\gamma})-\frac{1}{2}(y^2-\left(\frac{1}{1-\gamma}\right)^2)t_0}dy\right)=1,$$

where we used that $a<\frac{1}{1-\gamma}$ and, for the third term, the monotone convergence theorem. Therefore, for each $t_0\geq 0$,

$$\lim_{x\uparrow\infty}\frac{h(x,t_0)}{f\left(x,t_0\right)}=1. \tag{101}$$

We now use a result on the inverses of asymptotic functions (see [7]) to prove the limit in (100) by verifying the necessary assumptions for this result to hold. To this end, consider the function $g(z):=(1-\gamma)\ln z$, and notice that

$$g(f(z,t_0))=-\left(1-\gamma\right)\ln z+z-\frac{1}{2\left(1-\gamma\right)}t_0\sim z,\quad\text{as}\quad z\uparrow\infty.$$

Thus, $\lim_{z\uparrow\infty}z^{-1}g(f(z,t_0))=1$. Since, on the other hand, $\lim_{z\uparrow\infty}f(z,t_0)=\infty$, we deduce that $f^{(-1)}(x,t_0)\sim g(x)$, as $x\uparrow\infty$. Moreover, $g(x)$ is strictly increasing and the ratio $\frac{g_x(x,t_0)}{g(x,t_0)}\sim\frac{1}{x\ln x}=O(\frac{1}{x})$, for sufficiently large x. It then follows from the aforementioned result that $g(x)\sim h^{(-1)}(x,t_0)$, as $x\uparrow\infty$, and (100) follows.

Next, we claim that, for each $t_0\geq 0$,

$$\lim_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x\ln x}=1-\gamma. \tag{102}$$

Indeed, for $t_0=0$, we have from (94) that

$$x=\int_a^{\frac{1}{1-\gamma}}e^{yh^{(-1)}(x,0)}dy=\frac{1}{h^{(-1)}(x,0)}\left(e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}-e^{ah^{(-1)}(x,0)}\right), \tag{103}$$

and (100) yields that

$$\lim_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}}{x\ln x}$$

$$= \lim_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)}}{e^{\frac{1}{1-\gamma}h^{(-1)}(x,0)} - e^{ah^{(-1)}(x,0)}} \frac{h^{(-1)}(x,0)}{\ln x} = 1-\gamma.$$

For $t_0 > 0$, we deduce from (94) that

$$x = \frac{1}{\sqrt{t_0}} e^{-\frac{h^{(-1)}(x,t_0)^2}{2t_0}} \left(\Phi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} \right) - \Phi\left(a\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} \right) \right). \tag{104}$$

Then, using (97), we have, for large x,

$$\begin{aligned} 1 &\le \frac{1}{x\sqrt{t_0}} \exp\left(\frac{h^{(-1)}(x,t_0)^2}{2t_0} \right) \Phi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} \right) \\ &\le \frac{1}{x\sqrt{t_0}} e^{\frac{h^{(-1)}(x,t_0)^2}{2t_0}} \frac{1}{\frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} - \frac{1}{1-\gamma}\sqrt{t_0}} \varphi\left(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}} \right) \\ &= \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0) - \frac{1}{1-\gamma}t_0)}, \end{aligned}$$

and, in turn,

$$\begin{aligned} 1 &\le \liminf_{x\uparrow\infty} \left(\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \frac{h^{(-1)}(x,t_0)}{h^{(-1)}(x,t_0) - \frac{1}{1-\gamma}t_0} \right) \\ &= \liminf_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)} \lim_{x\uparrow\infty} \frac{h^{(-1)}(x,t_0)}{h^{(-1)}(x,t_0) - \frac{1}{1-\gamma}t_0} \\ &= \liminf_{x\uparrow\infty} \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}. \end{aligned} \tag{105}$$

Similarly, we use that, for $a < b < 0$,

$$\Phi(b) - \Phi(a) \ge \frac{\varphi(a) - \varphi(b)}{a}, \tag{106}$$

and deduce from (104) that, for large x,

$$\begin{aligned} 1 &\ge \frac{1}{x\sqrt{t_0}} e^{\frac{h^{(-1)}(x,t_0)^2}{2t_0}} \frac{1}{a\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}} \\ &\times \left(\varphi(a\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}) - \varphi(\frac{1}{1-\gamma}\sqrt{t_0} - \frac{h^{(-1)}(x,t_0)}{\sqrt{t_0}}) \right) \\ &= \frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0) - \frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0) - at_0)} - \frac{e^{a(h^{(-1)}(x,t_0) - \frac{1}{2}at_0)}}{x(h^{(-1)}(x,t_0) - at_0)}. \end{aligned}$$

For the second term, we have

$$\lim_{x\uparrow\infty} \frac{e^{ah^{(-1)}(x,t_0)}}{x(h^{(-1)}(x,t_0) - at_0)} e^{-\frac{1}{2}at_0} = \lim_{x\uparrow\infty} \frac{e^{ah^{(-1)}(x,t_0) - \ln x}}{h^{(-1)}(x,t_0) - at_0} e^{-\frac{1}{2}at_0}$$

$$= \lim_{x\uparrow\infty} \exp\left(a\ln x\left(\frac{h^{(-1)}(x,t_0)}{\ln x}-\frac{1}{a}\right)\right)\frac{1}{h^{(-1)}(x,t_0)-at_0}=0.$$

Therefore,

$$\limsup_{x\uparrow\infty}\left(\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0)-at_0)}-\frac{e^{a(h^{(-1)}(x,t_0)-\frac{1}{2}at_0)}}{x(h^{(-1)}(x,t_0)-at_0)}\right)$$

$$= \limsup_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0)-at_0)}-\lim_{x\uparrow\infty}\frac{e^{a(h^{(-1)}(x,t_0)-\frac{1}{2}at_0)}}{x(h^{(-1)}(x,t_0)-at_0)}$$

$$= \limsup_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{x(h^{(-1)}(x,t_0)-at_0)}$$

$$= \limsup_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}\lim_{x\uparrow\infty}\frac{xh^{(-1)}(x,t_0)}{x(h^{(-1)}(x,t_0)-at_0)}$$

$$= \limsup_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}\leq 1. \tag{107}$$

From (105) and (107), we then obtain

$$\limsup_{x\uparrow\infty}\frac{e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2}\frac{1}{1-\gamma}t_0)}}{xh^{(-1)}(x,t_0)}=1,$$

which together with (105) gives (102). Taking the logarithm of both sides then yields

$$\lim_{x\uparrow\infty}\left(\frac{1}{1-\gamma}\left(h^{(-1)}(x,t_0)-\frac{1}{2(1-\gamma)}t_0\right)-\ln x-\ln\ln x\right)=\ln(1-\gamma),$$

and the spatial asymptotic expansion (99) follows.

6.2.3 Spatial asymptotics of $r(x,t_0)$ for large x

Let $t_0>0$. We show that as $x\uparrow\infty$, the spatial asymptotic expansion of $r(x,t_0)$ is given by

$$\begin{aligned} r(x,t_0) &= \frac{1-\gamma}{t_0}x\ln\ln x+\frac{1}{t_0}\left((1-\gamma)x\ln x\right)^{a(1-\gamma)}e^{\frac{1}{2}a(\frac{1}{1-\gamma}-a)t_0} \\ &\quad +\frac{1}{2(1-\gamma)}x-\frac{1-\gamma}{t_0}x\ln\frac{1}{1-\gamma}+o(1). \end{aligned} \tag{108}$$

Indeed, from (21) and (93), we have

$$r(x,t_0)=\int_a^{\frac{1}{1-\gamma}}\frac{1}{t_0}(yt_0-h^{(-1)}(x,t_0))e^{yh^{(-1)}(x,t_0)-\frac{1}{2}y^2t_0}dy$$

$$+\frac{h^{(-1)}(x,t_0)}{t_0}\int_a^{\frac{1}{1-\gamma}} e^{yh^{(-1)}(x,t_0)-\frac{1}{2}y^2t_0}dy$$

$$=\frac{1}{t_0}\left(e^{a(h^{(-1)}(x,t_0)-\frac{1}{2}at_0)}-e^{\frac{1}{1-\gamma}(h^{(-1)}(x,t_0)-\frac{1}{2(1-\gamma)}t_0)}\right)+\frac{h^{(-1)}(x,t_0)}{t_0}x,$$

where we used (50) for the last term. Then, (108) follows using (99).

For $t_0=0$, we have from (103) that

$$r(x,0)=\frac{1}{1-\gamma}x-\frac{(\frac{1}{1-\gamma}-a)e^{ah^{(-1)}(x,0)}-x}{h^{(-1)}(x,0)},$$

and, for large x,

$$r(x,0)=\frac{1}{1-\gamma}x\left(1-\frac{1}{\ln x}\right)+o(1). \tag{109}$$

From (108) and (109), we then obtain that for $t_0>0$ and $t_0=0$, we have respectively,

$$r(x,t_0)\sim\frac{1-\gamma}{t_0}x\ln\ln x \qquad \text{and} \qquad r(x,0)\ \sim\ \frac{1}{1-\gamma}x.$$

Therefore, the risk tolerance function does *not* have the spatial turnpike property (37). Recall that the underlying measure *lacks* a Dirac mass on the right boundary of the measure μ, which is a necessary condition for the results in Proposition 3 to hold.

- The case $a=0^+$

We conclude with the case that μ is the Lebesgue measure on $(0,\frac{1}{1-\gamma}]$. For $t_0\geq 0$, we easily obtain the same spatial asymptotic expansions of $h^{(-1)}(x,t_0)$ as in (99) and of $r(x,t_0)$ as in (108) and (109).

For the temporal expansion, we claim that as $t\uparrow\infty$,

$$\frac{h^{(-1)}(x_0,t)}{t}=\frac{\sqrt{\ln t+2\ln x_0-\ln 2\pi}}{\sqrt{t}}+o(\frac{1}{\sqrt{t}}). \tag{110}$$

To see this, first recall (cf. (50)) that

$$x_0=\int_{0^+}^{\frac{1}{1-\gamma}} e^{y(h^{(-1)}(x_0,t)-\frac{1}{2}yt)}dy, \tag{111}$$

Taking the logarithm of both sides of (111) yields

$$2\ln x_0=\left(\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)^2-\ln t \tag{112}$$

$$+2\ln\left(\Phi\left(\sqrt{t}(\frac{1}{1-\gamma}-\frac{h^{(-1)}(x_0,t)}{t})\right)-\Phi\left(-\frac{h^{(-1)}(x_0,t)}{\sqrt{t}}\right)\right).$$

Next, we claim that $l := \liminf_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{\sqrt{t}} = \infty$. Indeed, if $l < \infty$, then, as $t \uparrow \infty$, the above yields

$$2\ln x_0 = l^2 - \lim_{t\uparrow\infty} (\ln t) + 2\ln(1 - \Phi(-l)) = -\infty,$$

which is a contradiction. Therefore, it must be that $l = \infty$, which combined with the fact that $\lim_{t\uparrow\infty} \frac{h^{(-1)}(x_0,t)}{t} = 0$, implies that as $t \uparrow \infty$, the third term on the right hand side of (112) converges to $2\ln\sqrt{2\pi}$. Thus, we obtain

$$2\ln x_0 = \lim_{t\uparrow\infty} \left(\left(\frac{h^{(-1)}(x_0,t)}{\sqrt{t}} \right)^2 - \ln t + 2\ln\sqrt{2\pi} \right),$$

from which we deduce that $h^{(-1)}(x_0,t) = \sqrt{t(\ln t + 2\ln x_0 - \ln 2\pi)} + o(\sqrt{t})$, and (110) follows.

7 Extensions

We have analyzed the spatial and temporal asymptotic behavior of the risk tolerance function $r(x,t)$. We recall that the optimal portfolio process $\pi_t^{*,x}$ is given in the feedback form $\pi_t^{*,x} = \sigma_t^+ \lambda_t r\left(X_t^{*,x}, t\right)$, with $X_t^{*,x}$ being the wealth generated by it. Furthermore, it was shown in [18] that $X_t^{*,x}$ and $\pi_t^{*,x}$ are given in the closed form

$$X_t^{*,x} = h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right), \ \pi_t^{*,x} = \sigma_t^+ \lambda_t h_x \left(h^{(-1)}(x,0) + A_t + M_t, A_t\right).$$

It is then natural to investigate the long-term limits $\lim_{t\uparrow\infty} X_t^{*,x}$, $\lim_{t\uparrow\infty} \pi_t^{*,x}$ under asymptotic assumptions on the initial datum and the results obtained herein. The asymptotic behavior of these processes has been investigated in [9] for the classical setting.

In a different direction, an interesting problem is how to construct investment policies which yield a targeted long-term wealth distribution. In a static model, this question was analyzed in [22] and in the log-normal, classical and forward cases, in [13]. However, in these settings, there is a strong model commitment, which is a nonrealistic assumption for long-term portfolio management.

In the forward setting we have analyzed herein, the model is dynamically updated. Furthermore, the distribution of the optimal wealth is given explicitly, using the above formula, by

$$\begin{aligned}\mathbb{P}\left(X_t^{*,x} \le y\right) &= \mathbb{P}\left(h^{(-1)}(x,0) + A_t + M_t \le h^{(-1)}(y, A_t)\right) \\ &= \mathbb{P}\left(\frac{h^{(-1)}(x,0)}{\langle M\rangle_t} + 1 + \frac{M_t}{\langle M\rangle_t} \le \frac{h^{(-1)}(y, A_t)}{\langle M\rangle_t}\right),\end{aligned}$$

where we used that $A_t = \langle M\rangle_t$ (cf. (15)). Therefore, one expects that the limit (53) as well as results on strong law of large numbers for martingales can be used to study the long-term distribution of the optimal processes. Such questions are currently investigated by the authors in [8] and others.

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