1. Find a power series representation for the function $f(x) = \arctan(x)$.

Hint: $\int \frac{1}{1+x^2} dx = \arctan(x) + C$.

Solution.

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ \frac{1}{1-(-x^2)} &= \sum_{n=0}^{\infty} (-x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{2n} \end{aligned}$$

Integrate term by term:

$$\int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

Important note: the above is just general anti-derivative of $\frac{1}{1+x^2}$. To find $\arctan(x)$, we need to determine the constant C.

Plug in $x = 0$: $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C = 0 + C = C$, $\arctan(0) = 0$, so $C = 0$. The conclusion is

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

2. Find a power series representation for the function $f(x) = ln(6-x)$.

Hint: You can approach it in two ways:

(1) Use the power series expansion for $\frac{1}{6-x}$ and integrate term by term. Do not forget the '+C'!

(2)OR: use the Taylor Series for $ln(1+x)$ at 0: $ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

Solution (1).

$$\frac{1}{6-x} = \frac{1}{6(1-\frac{x}{6})} = \frac{1}{6} \sum_{n=0}^{\infty} (\frac{x}{6})^n = \frac{1}{6} \sum_{n=0}^{\infty} \frac{1}{6^n} x^n$$

Notice that $\mathbf{\frac{d}{dx}(\ln(6-x)) = \frac{1}{6-x} \cdot (-1)}$, then $\ln(6-x) = \int -\frac{1}{6-x} dx + C$.
we integrate the series term by term:

$$\int -\frac{1}{6-x} dx = \int -\frac{1}{6} \sum_{n=0}^{\infty} \frac{1}{6^n} x^n dx = -\frac{1}{6} \sum_{n=0}^{\infty} \frac{1}{6^n} \frac{x^{n+1}}{n+1} + C$$

To determine the constant C, we plug in $x = 0$: $\ln(6-x) = \ln 6$, $-\frac{1}{6} \sum_{n=0}^{\infty} \frac{1}{6^n} \frac{0^{n+1}}{n+1} + C = C$, so $C = \ln 6$,

$$\begin{aligned} \ln(6-x) &= -\frac{1}{6} \sum_{n=0}^{\infty} \frac{1}{6^n} \frac{x^{n+1}}{n+1} + \ln 6 \\ &= \ln 6 - \sum_{n=0}^{\infty} \frac{1}{6^{n+1}} \frac{x^{n+1}}{n+1} \\ &= \ln 6 - \sum_{n=1}^{\infty} \frac{1}{6^n} \frac{x^n}{n} \\ &= \ln 6 - \frac{1}{6} x - \frac{1}{6^2 \cdot 2} x^2 - \frac{1}{6^3 \cdot 3} x^3 - \cdots \end{aligned}$$

Solution (2).

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\begin{aligned}
\ln(6-x) &= \ln\left(6(1-\frac{x}{6})\right) \\
&= \ln 6 + \ln\left(1+(-\frac{x}{6})\right) \\
&= \ln 6 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}(\frac{-x}{6})^n \\
&= \ln 6 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{(-1)^n}{6^n}x^n \\
&= \ln 6 + \sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{n6^n}x^n \\
&= \ln 6 + \sum_{n=1}^{\infty}\frac{(-1)}{n6^n}x^n \\
&= \ln 6 - \frac{1}{6}x - \frac{1}{6^2\cdot 2}x^2 - \frac{1}{6^3\cdot 3}x^3 - \cdots
\end{aligned}
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