1. (5 pts) If $g(x) = \frac{d}{dx}\left(\int_{2x}^{x^{\frac{1}{3}}} \frac{1}{1+t}dt\right)$, find $g(1)$.

Solution.

Let $f(u) = \frac{1}{1+t}$, $F(u)$ be the anti-dervative for f, i.e. $F'(u) = f(u)$.
Then by the Fundamental Theorem of Calculus and Chain Rule,

$$\begin{aligned}
\int_{2x}^{x^{\frac{1}{3}}} \frac{1}{1+t}dt &= F(x^{\frac{1}{3}}) - F(2x) \\
g(x) = \frac{d}{dx}\left(\int_{2x}^{x^{\frac{1}{3}}} \frac{1}{1+t}dt\right) &= F'(x^{\frac{1}{3}}) \cdot \frac{1}{3}x^{-\frac{2}{3}} - F'(2x) \cdot 2 \\
&= f(x^{\frac{1}{3}}) \cdot \frac{1}{3}x^{-\frac{2}{3}} - f(2x) \cdot 2 \\
&= \frac{1}{1+x^{\frac{1}{3}}} \cdot \frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{1+(2x)} \cdot 2 \\
g(1) &= \frac{1}{1+1} \cdot \frac{1}{3} \cdot 1 - \frac{1}{1+2} \cdot 2 \\
&= \frac{1}{6} - \frac{2}{3} \\
&= -\frac{1}{2}
\end{aligned}$$

Note: if you directly compute the integral and differentiate: the anti-derivative of $\frac{1}{1+x}$ is $ln(|1+x|)$,

$$\int_{2x}^{x^{\frac{1}{3}}} \frac{1}{1+t}dt = \ln|1+x^{\frac{1}{3}}| - \ln|1+2x|$$

$$g(x) = \frac{d}{dx}\left(\ln|1+x^{\frac{1}{3}}| - \ln|1+2x|\right) = \frac{1}{1+x^{\frac{1}{3}}} \cdot \frac{1}{3}x^{\frac{-2}{3}} - \frac{1}{1+(2x)} \cdot 2$$

And you still get the same result.

2. (5 pts) Find the value of $F''(0)$ when

$$F(x) = \int_5^x e^{\cos t} \sin t dt$$

Hint: $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$.

Solution.

$$\begin{aligned}
F'(x) &= e^{\cos x} \sin x \\
F''(x) &= e^{\cos x}(-\sin x)\sin x + e^{\cos x}\cos x \\
F''(0) &= e^1 \cdot (-0) \cdot 0 + e^1 \cdot 1 \\
&= e
\end{aligned}$$

Note: you can take the anti-derivative by substitution rule (will learn in Section 5.5): take $u = \cos t$, $du = -\sin t dt$,

$$\begin{aligned} \int e^{\cos t} \sin t dt &= \int e^u (-1) du \\ &= -e^u \\ &= -e^{\cos t} + C \end{aligned}$$

And derive $F(x) = -e^{\cos x} - (-e^{\cos 5})$, $F'(x) = e^{\cos t} \sin t$.