1. (5 pts) Evaluate the integral

$$I = \int_0^1 \frac{3}{\sqrt{4-x^2}}dx$$

Hint: $\left(sin^{-1}x\right)' = \frac{1}{\sqrt{1-x^2}}$.

Solution.

$$\sqrt{4-x^2} = \sqrt{4(1-\frac{x^2}{4})} = \sqrt{4}\cdot\sqrt{1-\left(\frac{x}{2}\right)^2} = 2\sqrt{1-\left(\frac{x}{2}\right)^2}$$

So we set $u = \frac{x}{2}$, $x = 2u$, $dx = 2du$. The limits of integration of $u = \frac{x}{2}$ is 0 and $\frac{1}{2}$.

$$\begin{aligned} I &= \int_0^1 \frac{3}{\sqrt{4-x^2}}dx \\ &= \int_0^1 \frac{3}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}dx \\ &= \int_0^{\frac{1}{2}} \frac{3}{2\sqrt{1-u^2}}2du \\ &= 3\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}du \\ &= 3\arcsin u\Big|_{u=0}^{u=\frac{1}{2}} \end{aligned}$$

Recall that

$$\begin{aligned} \theta = \arcsin(x) &\quad \text{means} \quad \sin\theta = x, \theta\in[-\frac{\pi}{2},\frac{\pi}{2}]. \text{ Domain of } x \text{ is } [-1,1]. \\ \theta = \arccos(x) &\quad \text{means} \quad \cos\theta = x, \theta\in[0,\pi]. \text{ Domain of } x \text{ is } [-1,1]. \\ \theta = \arctan(x) &\quad \text{means} \quad \tan\theta = x, \theta\in(-\frac{\pi}{2},\frac{\pi}{2}). \text{ Domain of } x \text{ is } (-\infty,\infty). \end{aligned}$$

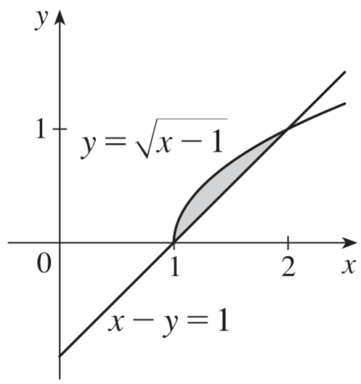
You can use a unit circle to find the arcsin values:$\arcsin(0) = 0$, $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$.

$$\begin{aligned} I &= 3\arcsin(\frac{1}{2}) - 3\arcsin(0) \\ &= 3\frac{\pi}{6} - 0 \\ &= \frac{\pi}{2} \end{aligned}$$

(Problem 2 on the back)

2. (5 pts) Find the area of the region bounded by the curves $y = \sqrt{x-1}$ and $x - y = 1$.

Solution.



$x - y = 1$ can also be expressed as $y = x - 1$. Graph the functions (see the above figure), find the intersections by

$$\begin{aligned} \sqrt{x-1} &= x-1 \\ x-1 &= (x-1)^2 \\ x-1 &= x^2-2x+1 \\ 0 &= x^2-3x+2 \\ 0 &= (x-1)(x-2) \\ x=1 &\text{ or } x=2 \end{aligned}$$

So the graphs intersect at $(1,0)$ and $(2,1)$. Use integral to calculate the area:

$$\begin{aligned} A &= \int_1^2 \sqrt{x-1} - (x-1)dx \\ &= \int_0^1 \sqrt{u} - udu \text{ by } u = x-1 \\ &= \frac{u^{\frac{3}{2}}}{(\frac{3}{2})} - \frac{u^2}{2}\Big|_{u=0}^{u=1} \\ &= \frac{2u^{\frac{3}{2}}}{3} - \frac{u^2}{2}\Big|_{u=0}^{u=1} \end{aligned}$$

$$= (\frac{2}{3} - \frac{1}{2}) - (0 - 0)$$
$$= \frac{1}{6}$$

You can also write $y = \sqrt{x-1}$ as $x = y^2 + 1$, write $x - y = 1$ as $x = y + 1$, solve for the intersection w.r.t. y:

$$\begin{aligned} y^2 + 1 &= y + 1 \\ y^2 - y &= 0 \\ y = 0 \quad &\text{or} \quad y = 1 \end{aligned}$$

$$\begin{aligned} A &= \int_0^1 (y+1) - (y^2+1) dy \\ &= \int_0^1 y - y^2 dy \\ &= \frac{y^2}{2} - \frac{y^3}{3}\Big|_{y=0}^{y=1} \\ &= (\frac{1}{2} - \frac{1}{3}) - (0 - 0) \\ &= \frac{1}{6} \end{aligned}$$